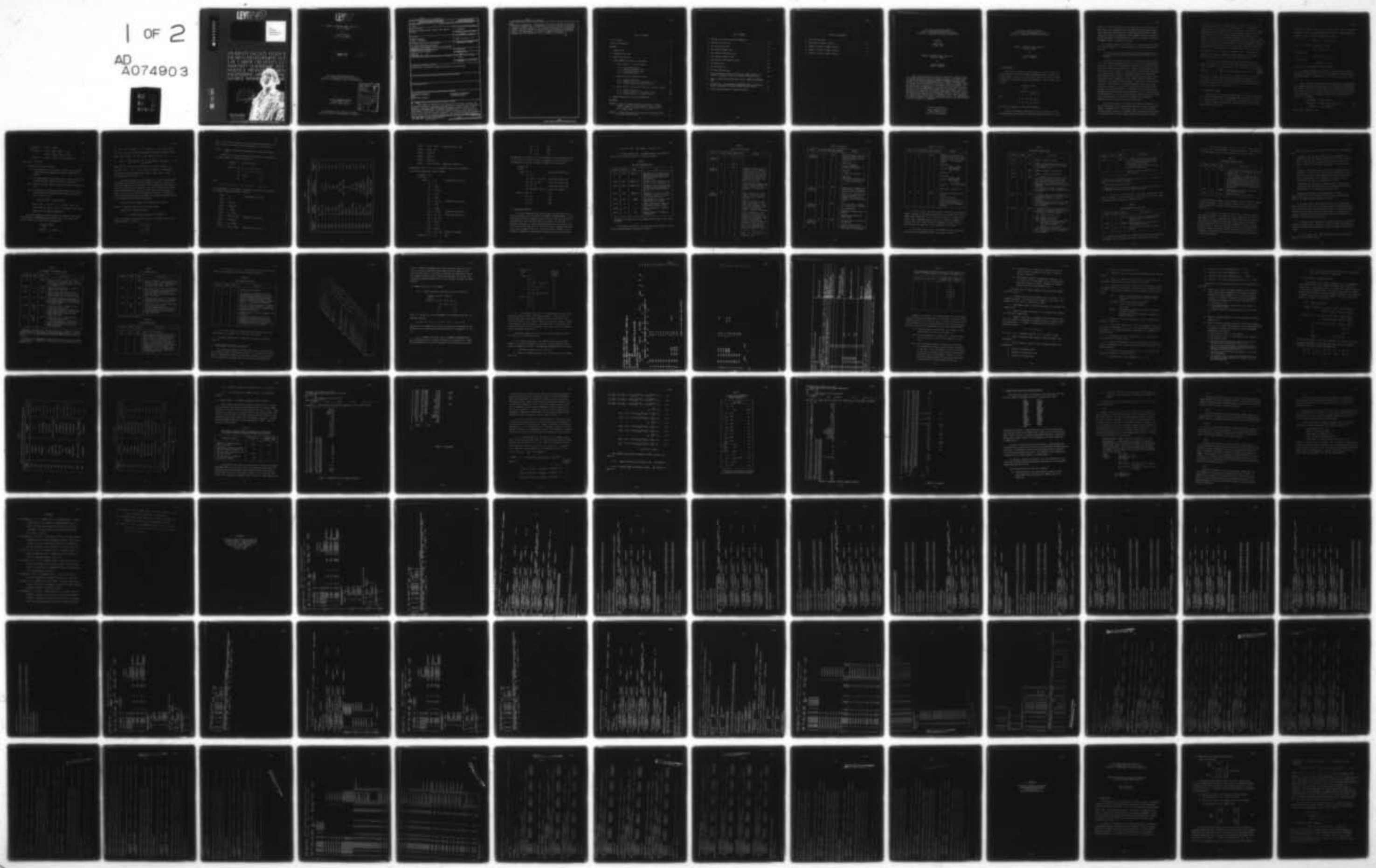
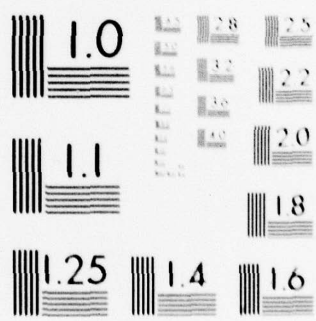
AD-A074 903 GEORGE WASHINGTON UNIV WASHINGTON D C INST FOR MANAG--ETC F/G 12/1
SYMBOLIC FACTORABLE SUMT; WHAT IS IT, HOW IS IT USED, (U)
MAY 79 A GHAEMI, G P MCCORMICK N00014-75-C-0611
UNCLASSIFIED SERIAL-T-402 NL

1 OF 2
AD
A074903




1.0
2.8
2.5
3.2
2.2
3.6
4.0
2.0
1.1
1.8
1.25
1.4
1.6

LEVEL IV

THE
GEORGE
WASHINGTON
UNIVERSITY

STUDENTS FACULTY STUDY R
ESEARCH DEVELOPMENT FUT
URE CAREER CREATIVITY CO
MMUNITY LEADERSHIP TECH
NOLOGY FRONTI ESIGN
ENGINEERING APP ENC
GEORGE WASHIN NIV

INSTITUTE FOR MANAGEMENT
SCIENCE AND ENGINEERING
SCHOOL OF ENGINEERING
AND APPLIED SCIENCE

LEVEL II

6 SYMBOLIC FACTORABLE SUMT: WHAT IS IT.
HOW IS IT USED,

by

10 Abolfazl Ghaemi
Garth P. McCormick

14 Serial-T-402
25 May 1979

12 112

11 25 may 79

The George Washington University
School of Engineering and Applied Science
Institute for Management Science and Engineering

Accession For		
NTIS GRA&I		☑
DDC TAB		☐
Unannounced		☐
Justification		
By		
Distribution/		
Availability Codes		
Dist	Avail and/or special	
A		

15 Contract N00014-75-C-0611
Project NR 047 142
Office of Naval Research

406 743

NONE

SECURITY CLASSIFICATION OF THIS PAGE (*When Data Entered*)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER T-402	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (*and Subtitle*) SYMBOLIC FACTORABLE SUMT: WHAT IS IT? HOW IS IT USED?		5. TYPE OF REPORT & PERIOD COVERED SCIENTIFIC 6. PERFORMING ORG. REPORT NUMBER T-402
7. AUTHOR(s) ABOLFAZL GHAEMI GARTH P. MCCORMICK		8. CONTRACT OR GRANT NUMBER(s) N00014-75-C-0611
9. PERFORMING ORGANIZATION NAME AND ADDRESS THE GEORGE WASHINGTON UNIVERSITY INSTITUTE FOR MANAGEMENT SCIENCE AND ENGINEERING WASHINGTON, D.C. 20052		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS OFFICE OF NAVAL RESEARCH CODE 434 ARLINGTON, VA 22217		12. REPORT DATE 25 May 1975 13. NUMBER OF PAGES 107
14. MONITORING AGENCY NAME & ADDRESS(*if different from Controlling Office*)		15. SECURITY CLASS. (*of this report*) NONE 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (*of this Report*)

DISTRIBUTION IS UNLIMITED; APPROVED FOR PUBLIC SALE AND RELEASE

17. DISTRIBUTION STATEMENT (*of the abstract entered in Block 20, if different from Report*)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (*Continue on reverse side if necessary and identify by block number*)

NONLINEAR PROGRAMMING
SUMT
SENSITIVITY ANALYSIS
FACTORABLE PROGRAMMING
OPTIMIZATION CODE

20. ABSTRACT (*Continue on reverse side if necessary and identify by block number*)

This report is intended to serve as a guide for coding, solving, and conducting sensitivity and elasticity analysis on a solution vector of general parametric nonlinear programming problems using the latest version of the computer programming model of SYMBOLIC FACTORABLE SUMT. The motivation for development of this model, which uses symbolic factorable input, and the process of its evolution is briefly reviewed. The relative merits and advantages of this model over its earlier versions, as well as over the SUMT-Version 4 (Cont'd) over

model [1] is elaborated. The procedure for putting problems into symbolic factorable language that is digestible by the model, and the related model outputs, are fully explored. For illustrative purposes three increasingly difficult sample problems are coded, solved, and discussed in sufficient detail. For completeness, the theory of symbolic factorable sensitivity analysis that was developed in [4] is included as an appendix.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Management Science and Engineering

Abstract
of
Serial T-402
25 May 1979

SYMBOLIC FACTORABLE SUMT: WHAT IS IT?
HOW IS IT USED?

by

Abolfazl Ghaemi
Garth P. McCormick

This report is intended to serve as a guide for coding, solving, and conducting sensitivity and elasticity analysis on a solution vector of general parametric nonlinear programming problems using the latest version of the computer programming model of SYMBOLIC FACTORABLE SUMT. The motivation for development of this model, which uses symbolic factorable input, and the process of its evolution is briefly reviewed. The relative merits and advantages of this model over its earlier versions, as well as over the SUMT-Version 4 model [1] is elaborated. The procedure for putting problems into symbolic factorable language that is digestible by the model, and the related model outputs, are fully explored. For illustrative purposes three increasingly difficult sample problems are coded, solved, and discussed in sufficient detail. For completeness, the theory of symbolic factorable sensitivity analysis that was developed in [4] is included as an appendix.

Research supported by

Contract N00014-75-C-0611
Project NR 047 142
Office of Naval Research

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Management Science and Engineering

SYMBOLIC FACTORABLE SUMT: WHAT IS IT? HOW IS IT USED?

by

Abolfazl Ghaemi
Garth P. McCormick

1. Introduction

This report is intended to guide individuals working in the area of optimization in the use of SYMBOLIC FACTORABLE SUMT, a computer programming model, to code, solve, and conduct sensitivity analysis studies on parametric mathematical optimization problems of the following general structure:

$$\begin{aligned} &\underset{x \in E^n}{\text{minimize}} \quad F(x,y) \\ &\text{subject to} \quad G(x,y) \geq 0 \\ &\qquad\qquad\quad H(x,y) = 0 \, , \end{aligned}$$

where

$$G: \; E^n \times E^K \rightarrow E^m \quad \text{and} \quad C^2$$

$$H: \; E^n \times E^K \rightarrow E^p \quad \text{and} \quad C^2$$

and $y \in E^K$ corresponds to the problem parameters.

Since its development, SUMT-Version 4 [1] has undergone a series of modifications and refinements; SYMBOLIC FACTORABLE SUMT is its latest

version. The primary incentive for such modifications has been to relax the coding of the problem functions and their first and second derivatives, required by SUMT-Version 4 as users' subroutines, which in many cases have proved to be cumbersome, time consuming, costly, and prone to various derivational and coding errors. Table 13, provided in Section 4 for sample example 3, supports this claim.

The computer program reported here is an outgrowth of the program developed by McCormick [2]. The rationale for such an attempt was the fact that

> "Almost every nonlinear function can systematically be decomposed to (and hence synthesized from) simple functions of one variable."

Using this fact, which in essence constitutes the backbone structure of factorable programming--the topic of the next section--McCormick developed a series of subroutines which internally calculated the values and the first and second derivatives of single variable functions, many of which are listed in Table 12. Each of these functions was associated with a numeric code, called a transformation code, which aided their identification in the routines. Another subroutine was provided to synthesize any desired combination of these single variable functions to reconstruct the intended problem functions. Interface of these routines with those of SUMT-Version 4 relaxed the provision of the user's subroutines to the model. In fact, the user's task boiled down to decomposition of his problem functions to functions of single variables and earmarking them with appropriate transformation codes. This was a major breakthrough which to a great extent alleviated the user's task in coding and solving optimization problems, and brought about significant savings of the time and cost involved in these steps.

Yet another attempt to facilitate the coding of problems was by deSilva and McCormick [3], through which evolved the original version of SYMBOLIC FACTORABLE SUMT. The essential task in [3] was to associate relevant symbolic names with individual transformation codes which aided the identification of single variable functions in the model software.

Furthermore, this version permitted symbolic input for the problem variables and parameter names of up to four characters. In addition, it simplified the coding of the problems so that users could readily acquaint themselves with the symbolic names of various single variable functions, and could then code their problems without frequent reference to the manual. This was not the case in the earlier version, because the single variable functions had to be associated with the related transformation codes. Later, by exploiting the relative merits of factorable functions, deSilva and McCormick developed new routines to calculate a weighted cumulative sensitivity of the components of a solution vector with respect to problem parameters $\underset{\sim}{\sigma}^T V_y x^*$ [Appendix 2, Equation (37)]. The underlying theory for the above developments is fully explained in [4], which is included in Appendix 2 for reference by the interested user.

This latter version has recently been further modified and refined by the authors to calculate sensitivity information for the individual components of the solution vector x^* with respect to the problem parameters y (i.e., $\frac{dx_i^*}{dy_j}$) and corresponding elasticities (i.e., $\frac{dx_i^*/x_i^*}{dy_j^*/y_j^*}$) for $i=1,\ldots,n$; $j=1,\ldots,k$. In order to enhance its scope, a few more functions of a single variable have been added to the model. This latest version of the model is compiled at The George Washington University IBM/370, and can be accessed with the aid of job control cards, as depicted in Figures 2 or 3 of Section 4.

2. Factorable Functions

Most functions of several variables used in nonlinear optimization are complicated compositions of transformed sums and products of functions of a single variable. Once this observation is made formal, an outline of an automatic computer-oriented way of representing nonlinear programming problems becomes clear.

Definition: A function $f(x_1,\ldots,x_n)$ is a *factorable function* of several variables if it can be represented as the last in a finite sequence of functions $f^j(x)$, which are composed as follows:

$$f^j(x) = x_j \ , \quad \text{for } j=1,\ldots,n \ .$$

For $n < j$, $f^j(x)$ is either of the form

$$f^k(x) + f^\ell(x) \ , \quad k,\ell < j \ ,$$

or of the form

$$f^k(x) \cdot f^\ell(x) \ , \quad k,\ell < j \ ,$$

or of the form

$$T[f^k(x)] \ , \quad k < j \ ,$$

where $T(f)$ is a function of a single variable.

It should be emphasized that this is a *natural* way of looking at functions of several variables, since it corresponds to the way they are used when evaluated at a particular set of values of the vector $(x_1,\ldots,x_n)$.

When a function is represented in factorable form, the computation of its gradient vector and Hessian matrix of second derivatives becomes very simple. Consider the general form for the gradient and Hessian matrix of a function which is either the sum of two functions, the product of two functions, or a transformation (function of one variable) of a function. For example,

$$\nabla[f(x)+g(x)] = \nabla f(x) + \nabla g(x)$$

$$\nabla[f(x)\cdot g(x)] = \nabla f(x) \cdot g(x) + \nabla g(x) \cdot f(x) \tag{1}$$

$$\nabla T[f(x)] = \nabla f(x) \cdot \dot{T}[f(x)] \ , \tag{2}$$

where $\dot{T}(f) = \partial T(f)/\partial f$; and

$$\nabla^2[f(x)+g(x)] = \nabla^2 f(x) + \nabla^2 g(x)$$

$$\nabla^2[f(x)\cdot g(x)] = \nabla^2 f(x)\cdot g(x) + \nabla^2 g(x)\cdot f(x) + \nabla f(x)\cdot\nabla^T g(x) + \nabla g(x)\cdot\nabla^T f(x)\ , \quad (3)$$

$$\nabla^2[T[f(x)]] = \nabla^2 f(x)\cdot\dot{T}[f(x)] + \nabla f(x)\ddot{T}[f(x)]\nabla^T f(x)\ , \quad (4)$$

where $\ddot{T}(f) = \partial^2 T(f)/\partial f^2$.

Several points should be noted.

(i) The computation of (1) involves $f(x), g(x)$, which would already have been computed in order to evaluate their product.

(ii) The computation of (3) involves $\nabla f(x)$, $\nabla g(x)$, which would already have been computed in order to evaluate (1).

(iii) Equation (4) used $\dot{T}[f(x)]$, which was previously needed in (2).

(iv) By induction, it follows that Hessian matrices of factorable functions are naturally given as sums of outer products of vectors, i.e.,

$$\sum u_i(x)\alpha_i(x)v_i^T(x) + v_i(x)\alpha_i(x)u_i^T(x)\ ,$$

where $u_i(x)$, $v_i(x)$ are $n \times 1$ vectors, $\alpha_i(x)$ are scalars, and the $u_i(x)$, $v_i(x)$ are available, having been required for the gradient computation.

For those optimization problems which are factorable, the factorable nonlinear programming problem is written in a more compact form (often referred to as canonical factorable form):

$$\underset{x\in E^n}{\text{minimize}}\ X^N(x)$$

$$\text{subject to}\ L_i \leq X^i(x) \leq U_i\ ,$$

for $i=1,\ldots,N-1$ (possibly $L_i = -\infty$ and/or $U_i = +\infty$) , where $X^i(x) \equiv x_i$ (for $i=1,\ldots,n$) , and the remainder are defined recursively as follows: given $X^p(x)$, for $p=1,\ldots,i-1$, then for $i=n+1,\ldots,N$,

$$X^i(x) = \sum_{p=1}^{i-1} T_p^i[X^p(x)] + \sum_{p=1}^{i-1} \sum_{q=1}^{p} V_{q,p}^i[X^p(x)] \cdot U_{p,q}^i[X^q(x)] , \quad (5)$$

where the T's, U's, and V's are functions of a *single* variable. The functions $X^i(x)$, $i=1,\ldots,N$ are usually referred to as concomitant variable functions and abbreviated as CVFs. Also, $X^i(x)$, $i=n+1,\ldots,N$ are referred to as new CVFs.

An important point bears mentioning here. One of the many appealing features of coding and solving problems in factorable form is that they lend themselves in a natural way to the computation of the number of functional evaluations and arithmetic operations involved in the evaluation of problem functions, and in the first and second derivatives required for solution of nonlinear optimization problems. By doing so, a precise quantitative measure is provided for comparing algorithms devised for solving nonlinear optimization problems.

Following are two examples that demonstrate these ideas.

Example i [putting a function into factorable form].

Consider the following function:

$$W = \Gamma \cdot B(x_1+x_2)(\ell_t x_4+\ell_m x_5+\ell_b x_6)[x_1 + (x_2^2-x_3^2)^{\frac{1}{2}}]^{-1} ,$$

which corresponds to the objective function of the sample problem 3 provided in Section 4.3, where

$$\Gamma \cdot B = 3736.6$$
$$\ell_t = 495$$
$$\ell_m = 385$$
$$\ell_b = 315 .$$

Table 1 shows how this function is put into simple factorable form, and depicts the corresponding stepwise reconstruction of the function.

Example ii [putting an optimization problem into canonical factorable form].

Consider the following optimization problem, which corresponds to the sample example 1 provided in Section 4.1:

$$\underset{x_1,\,x_2}{\text{minimize}} \quad f = (x_1-10)^2 + (x_2-1)^2$$

$$\begin{aligned} \text{subject to} \quad & g_1: \; C_5 \; x_2 - 5 \sin .5x_1 \leq C_1 \\ & g_2: \; x_2 - C_6 \ln x_1 \geq C_2 \\ & g_3: \; x_1 x_2 \geq C_3 \\ & g_4: \; x_1 \geq C_4 , \end{aligned}$$

where

$$C_1 = C_2 = 0 , \quad C_3 = 3 , \quad C_4 = 1.5 , \quad C_5 = 1 , \quad C_6 = -1.5$$

are the parameters of the problem. Putting the functions of this problem into simple factorable form goes as follows:

$$\begin{aligned} X^1(x) &= x_1 && \text{(functional form of } g_4) \\ X^2(x) &= x_2 \\ X^3(x) &= .5X^1(x) \\ X^4(x) &= \text{Sin}[X^3(x)] \\ X^5(x) &= -\,5X^4(x) \\ X^6(x) &= C_5 \cdot X^2(x) \\ X^7(x) &= X^6(x) + X^5(x) && \text{(functional form of } g_1) \\ X^8(x) &= \ln\,[X^1(x)] \\ X^9(x) &= -C_6 \cdot X^8(x) \\ X^{10}(x) &= X^2(x) + X^9(x) && \text{(functional form of } g_2) \end{aligned}$$

TABLE 1

EXAMPLE OF FACTORABLE FUNCTION (EXAMPLE i)

Function in Sequence		Form of Composition	Portion of Original Function	Composition Type
$f^j(x)$	=	x_j , j=1,6		$\Gamma_j(x) = x_j$
$f^7(x)$	=	$f^1(x) + f^2(x)$	$x_1 + x_2$	f + g
$f^8(x)$	=	$\Gamma B \cdot f^7(x)$	$\Gamma B(x_1+x_2)$	T(f)
$f^9(x)$	=	$\ell_t \cdot f^4(x)$	$\ell_t x_4$	T(f)
$f^{10}(x)$	=	$\ell_m \cdot f^5(x)$	$\ell_m x_5$	T(f)
$f^{11}(x)$	=	$\ell_b \cdot f^6(x)$	$\ell_b x_6$	T(f)
$f^{12}(x)$	=	$f^9(x) + f^{10}(x)$	$\ell_t x_4 + \ell_m x_5$	f + g
$f^{13}(x)$	=	$f^{12}(x) + f^{11}(x)$	$\ell_t x_4 + \ell_m x_5 + \ell_b x_6$	f + g
$f^{14}(x)$	=	$[f^2(x)]^2$	x_2^2	T(f)
$f^{15}(x)$	=	$-[f^3(x)]^2$	$-x_3^2$	T(f)
$f^{16}(x)$	=	$f^{14}(x) + f^{15}(x)$	$x_2^2 - x_3^2$	f + g
$f^{17}(x)$	=	$[f^{16}(x)]^{.5}$	$(x_2^2 - x_3^2)^{.5}$	T(f)
$f^{18}(x)$	=	$f^1(x) + f^{17}(x)$	$x_1 + (x_2^2-x_3^2)^{.5}$	f + g
$f^{19}(x)$	=	$[f^{18}(x)]^{-1}$	$[x_1 + (x_2^2-x_3^2)^{.5}]^{-1}$	T(f)
$f^{20}(x)$	=	$f^8(x) \cdot f^{13}(x)$	$\Gamma B(x_1+x_2)(\ell_t x_4+\ell_m x_5+\ell_b x_6)$	$f \cdot g$
$f^{21}(x)$	=	$f^{20}(x) \cdot f^{19}(x)$	original function	$f \cdot g$

$$X^{11}(x) = X^1(x) \cdot X^2(x) \qquad \text{(functional form of } g_3\text{)}$$
$$X^{12}(x) = X^1(x) - 10$$
$$X^{13}(x) = [X^{12}(x)]^2$$
$$X^{14}(x) = X^2(x) - 1$$
$$X^{15}(x) = [X^2(x)-1]^2$$
$$X^{16}(x) = X^{13}(x) + X^{15}(x) \qquad \text{(objective function f)} .$$

Representing $X^i(x)$ by X_i for notational simplicity, one canonical form of this problem will read as follows:

$$\underset{X_1,\ldots,X_{15}}{\text{minimize}} \quad X_{16}$$

$$\begin{array}{llll}
\text{where} & X_1 = x_1 & & \text{(functional form of } g_4\text{)} \\
& X_2 = x_2 & & \\
& X_3 = .5X_1 & & \\
& X_4 = \text{Sin } X_3 & & \\
& X_5 = -5X_4 & & \\
& X_6 = C_5X_2 & & \\
& X_7 = X_6 + X_5 & & \text{(functional form of } g_1\text{)} \\
& X_8 = \ln X_1 & & \\
& X_9 = -C_6X_8 & & \\
& X_{10} = X_2 + X_9 & & \text{(functional form of } g_2\text{)} \\
& X_{11} = X_1 \cdot X_2 & & \text{(functional form of } g_3\text{)} \\
& X_{12} = X_1 - 10 & & \\
& X_{13} = (X_{12})^2 & & \\
& X_{14} = X_2 - 1 & & \\
& X_{15} = (X_{14})^2 & & \\
& X_{16} = X_{13} + X_{15} & & \text{(objective function)} \\
\text{subject to} & X_7 \leq C_1 & & (g_1)
\end{array}$$

$$
\begin{array}{lll}
X_{10} \geq C_2 & & (g_2) \\
X_{11} \geq C_3 & & (g_3) \\
X_1 \geq C_4 & & (g_4) .
\end{array}
$$

By exploiting the definition of the concomitant variable functions given in (5), many of the above steps may be combined and the problem may be reformulated (as is done in symbolic codings) more concisely as:

$$
\begin{array}{lll}
\underset{X \varepsilon E^6}{\text{minimize}} & X^6 & \\
\text{where} & X^1 = x_1 & \text{(functional form of } g_4) \\
 & X^2 = x_2 & \\
 & X^3 = C_5 X^2 - 5 \sin .5X^1 & \text{(functional form of } g_1) \\
 & X^4 = X^2 - C_6 \ \ell n \ X^1 & \text{(functional form of } g_2) \\
 & X^5 = X^1 \cdot X^2 & \text{(functional form of } g_3) \\
 & X^6 = (X^1-10)^2 + (X^2-1)^2 & \text{(objective function)} \\
\text{subject to} & X^3 \leq C_1 & (g_1) \\
 & X^4 \geq C_2 & (g_2) \\
 & X^5 \geq C_3 & (g_3) \\
 & X^1 \geq C_4 & (g_4) .
\end{array}
$$

3. Input Specifications

This section describes how the user must code and supply the required information about his problem to SYMBOLIC FACTORABLE SUMT. In the following, the type of information and its format, name, and description are discussed in the same order that must appear in a coded deck. A schematic depiction of a coded problem deck structure is provided in Figure 1 at the end of this section. For clarification, frequent reference to the code of sample problem 1 of Section 4.1 and to Figures 2 or 3 is advised for beginners as they proceed through the input specifications.

1 *JOB and JCL cards.* (See Figure 2, Section 4.1.4.)

2 *The SUMT parameter card.* The SUMT parameter card, which is read in by the subroutine MAIN, is described in Table 2.

TABLE 2

THE SUMT PARAMETER CARD

Columns	Format	Name	Use
01-12	E12.0	EPSI (ε)	Tolerance used to decide if an unconstrained minimum has been achieved for each subproblem (see Option 9)
13-24	E12.0	RHOIN (r_1)	Possible initial value of r (often set at 1.0) (see Option 1)
25-36	E12.0	THETAO (θ_0)	Tolerance used to decide if the solution to the NLP problem (A) has been approximated (see Option 5)
37-48	E12.0	RATIØ (c)	Parameter (> 1) used to compute consecutive values of r; $r_{i+1} = r_i/c$ (often set at 16.0)
49-60	E12.0	TMMAX	Maximum amount of time for solving problem (in seconds)
61-64	I4	M	Number (integer) of inequality constraints, (M+MZ) $\leq$ 200*
65-68	I4	N	Number (integer) of variables, N $\leq$ 100
69-72	I4	MZ	Number (integer) of equality constraints

*The limits on M+MZ and N are governed by the sizes of the arrays in SUMT.

3 *The first option card.* The first option card, which is read in by the subroutine MAIN, is described in Table 3.

TABLE 3

THE FIRST OPTION CARD

Option	Column	Format	Name	Value	Meaning
1 (normally set to 3)	7	I	NT1	3	
2	14	I	NT2	2	
3 (normally set to 1)	21	I	NT3	1	Standard printout (this includes a call to OUTPUT after the solution of every subproblem). Also the estimates of the "Lagrange multipliers" and first- and second-order solution estimates are printed.
				2	For additional printout (includes standard printout and every intermediate point, gradient of P, and the vector S).
4 (normally set to 1)	28	I	NT4	1	Final convergence is determined on the basis of current solution to the subproblem.
				2	Final convergence is determined on the basis of the first order estimates. The first order estimate of the solution vector must be close to feasible. See below.
				3	Final convergence is determined on the basis of the second order estimates. The second order estimate of the solution vector must be close to feasible before the convergence check is made. If $\overline{x}$ is a solution estimate it is considered close to being feasible if $g_1(x) + \theta_0 \geq 0$, $i=1,2,\ldots,m$, where θ_0 is defined on the parameter card.

TABLE 3--*continued*

Option	Column	Format	Name	Value	Meaning
5 (normally set to 1)	35	I	NT5		The convergence criterion determining the NLP problem has been solved (only use = 1 when NT4 $\neq$ 1)
				1	Quit when $\dfrac{G - f[x(r_k)]}{G[x(r_k),\mu(r_k),\lambda(r_k)]} < \theta_0$
				2	Quit when $\left\lvert r \sum_{j=1}^{m} \ln g_j[x(r_k)] \right\rvert < \theta_0$
				3	Quit when $\dfrac{\text{first order estimate of } v_0}{G[x(r_k),\mu(r_k),\lambda(r_k)]} - 1 < \theta_0$
6 (normally set to 1)	42	I	NT6	1	After final convergence the program reads in new data and solves the next problem
				2	After final convergence has been determined a call to PUNCH is made before proceeding on to the next problem
7 (normally set to 1)	49	I	NT7		First move after a minimum to a subproblem is achieved
				1	No extrapolation
				2	Extrapolate through last two minima
				3	Extrapolate through last three minima
8 (normally set to 0)	56	I	NT8	0	No sensitivity or elasticity calculation
				2	Conduct sensitivity and elasticity calculation after each subproblem convergence

TABLE 3--*continued*

Option	Column	Format	Name	Value	Meaning
9	63	I	NT9		Subproblem convergence criterion, or when to stop minimizing P function for fixed value of r (see parameter card)
				1	Quit when $\left\lvert \nabla_x P^T(x^i,r) \left[\frac{\partial^2 P(x,r)}{\partial x_i \partial x_j}\right]^{-1} \cdot \nabla_x P(x^i,r) \right\rvert < \varepsilon$
				2	Quit when $\lvert \nabla_x P^T(x^i,r) \left[\frac{\partial^2 P(x,r)}{\partial x_i \partial x_j}\right]^{-1} \cdot \nabla_x P(x^i,r) < \frac{P(x^{i-1}) - P(x^i)}{5}$
				3	Quit when $\lvert \nabla_x P(x^i,r) \rvert < \varepsilon$
10	70	I	NT10	1	At least one nonlinear constraint
				2	Linear constraints
				3	Linear constraints and linear objective function (i.e., a linear programming problem)

4 *The title card.* This card, which is read in by the subroutine SYMPUT, contains all pertinent information such as problem title and reference, user's name, date, etc., that the user wishes to appear on the output. This information must be coded in a single card anywhere in columns 1-80, and is read in with the alphanumeric format of 20 A4. A blank card must be supplied if this information is not coded.

5 *The SYMPUT parameter card.* The SYMPUT parameter card, which is read in by the subroutine SYMPUT, is described in Table 4.

TABLE 4

THE SYMPUT PARAMETER CARD

Columns	Format	Name	Description
01-05	I_5	N	Number of problem variables
06-10	I_5	NN	Number of concomitant variable functions, CVFs (i.e., number of variables in factorable form)
11-15	I_5	MMM	Number of inequality constraints
16-20	I_5	MZ	Number of equality constraints
21-25	I_5	ITEST	• If ITEST=0, at the initial feasible point of the problem the values and first and second derivatives of new CVFs will be evaluated and printed for debugging purposes, without solving the problem • If ITEST=1, the above information will be suppressed and the problem will be solved
26-30	I_5	IPR	• If IPR=1, after each subproblem convergence the values of all CVFs at the subproblem solution point will be printed. • If IPR=0 (≠1), the values of the CVFs will be suppressed after each subproblem convergence
31-35	I_5	ITES	• If ITES=2, after each subproblem convergence the following information will be printed for debugging purposes: (i) Entries of vector $\sigma^{(1)}$ (ii) Entries of vector $\alpha^{(2)}$ (iii) Values of all CVFs (iv) Partial derivatives of new CVFs with respect to problem variables and parameters (v) Entries of vector DELA$^{(3)}$ • If ITES=0 (≠2), the above information will be suppressed after each subproblem convergence

TABLE 4--*continued*

Columns	Format	Name	Description
36-40	I_5	NSWIT	• If NSWIT=1, the modified listing of the coded problem and related tables set up by subroutine SYMPUT will be suppressed in the output • If NSWIT=0 ($\neq$1), the above listing will not be suppressed
41-45	I_5	IID	Number of problem parameters (i.e., P and D cards)

(1) Vector σ is an n-dimensional unit vector e_i, i=1,...,n (n is the number of original problem variables).

(2) See Equation (38) in Appendix 2.

(3) Negative values of partial derivatives of subproblem solution vector components with respect to the problem parameters.

6 *The variable definition cards.* These cards are read in by the subroutine SYMPUT. For each variable of the problem, the user must supply one card with the descriptions as shown in Table 5.

TABLE 5

THE VARIABLE DEFINITION CARDS

Columns	Format	Name	Description
02	A1	ATYPE	Letter "v" must be coded for ATYPE to indicate this card corresponds to a variable card
04-07	A_4	ANAME	Variable name to be specified by the user (right justified)
16-28	F12.5	ARL	Initial value of variable (starting point)

It is obvious that the number of variable definition cards should match the number of the (original) variables of the problem being coded.

7 *The problem parameter cards (P and D cards)*. These cards are read in by the subroutine SYMPUT. For each parameter of the problem, the user must supply one card with a description as shown in Table 6.

TABLE 6

THE PROBLEM PARAMETER CARDS

Columns	Format	Name	Description
02	A_1	ATYPE	• Letter "D" must be coded for ATYPE to indicate this card corresponds to a parameter card • Use of letter "P" instead of letter "D", in conjunction with proper option in the SUMT parameter card, will invoke the sensitivity routines to calculate the sensitivity and elasticities of the solution point with respect to the indicated parameter after each subproblem convergence
04-07	A_4	ANAME	Parameter name to be specified by the user (right justified)
16-28	F12.5	ARG	Numerical value of the parameter

It is obvious that the number of the parameter cards must match the number of problem parameters. Also, parameter cards with "P", on which sensitivity calculation is desired, *must* appear before parameter cards with letter "D".

8 *The separable and quadratic cards*. These cards are read in by the subroutine SYMPUT. The number and order of appearance of separable and quadratic cards will depend on the problem functions being coded and their canonical formulation by the user. Each separable or quadratic card in essence provides the model with information regarding a single variable function. With this information, the model can systematically synthesize the new concomitant variable functions and subsequently can reconstruct the problem functions and their derivatives for the purposes of various calculations.

While coding these cards, the user must be aware of the following facts:

- Separable cards may appear sequentially in any numbers. As long as they are associated with the same new CVF (CVF in columns 04-07 of Table 7), the single variable functions in each card will be *summed up* recursively by the model to construct the new CVF.

- Quadratic cards must appear as pairs, with any number of pairs permitted. As long as these cards are associated with the same new CVF, the *product* of the related single variable functions in each consecutive pair will be *summed up* recursively (if there are more than one pair) to reconstruct the new CVF.

- When there are separable *and* quadratic cards associated with the same new CVF, the results of the aforementioned calculations will be *summed up* to reconstruct the subject CVF.

Separable and quadratic cards are described in Table 7. For further clarification, the reader is referred to the formulation and coding of the sample problems provided in Section 4.

9 The bound cards. These cards are read in by the subroutine SYMPUT. There are two types of bound cards, upper bound and lower bound. It is natural that these cards must appear after the concomitant variable functions that are being bounded have been defined. Cards relating to equality constraints *must* follow the cards defining the inequality constraints. Table 8 describes bound cards.

10 The flag card. The user must indicate the end of his problem's symbolic factorable code by furnishing a flag card. This card, which is read in by the subroutine SYMPUT, must contain the number 5 in its second column.

11 The tolerance card. This card is read in by the subroutine MAIN, and is described in Table 9.

TABLE 7

THE SEPARABLE AND QUADRATIC CARDS

Columns	Format	Name	Description
02	A1	ATYPE	Letter "S" must be coded for ATYPE if the card is a separable card. Letter "Q" must be coded for ATYPE if the card is a quadratic card
04-07	A4	ANAME	Name of the new CVF must be coded for ANAME (right justified)
09-12	A4	ARG	Name of a previously defined CVF, in terms of which the current new CVF is being defined, must be coded for ARG
14-16	A4	AFUN	Three-letter symbolic name assigned to the type of functional relationship between ANAME and ARG (see Table 12) must be coded for AFUN
17-28	F12.5 or 3A4	AC1†	• The value of C1 in AFUN (see Table 12) must be coded for AC1 • If C1 corresponds to a previously defined problem parameter, then the parameter name must be coded for AC1 (right justified)*
29-40	F12.5 or 3A4	AC2†	• The value of C2 in AFUN (see Table 12) must be coded for AC2 • If C2 corresponds to a previously defined problem parameter, then the parameter name must be coded for AC2 (right justified)*

†Reading of AC1 and/or AC2 with the proper F or A format in the model is made possible by utilizing the subroutine REREAD, which is a built-in subroutine at The George Washington University FORTRAN library (see Section 5).

*Parameters with alphanumeric value *cannot* take negative signs throughout the code (i.e., -ABC does not imply the negative of numeric value of parameter ABC).

TABLE 8

THE BOUND CARDS

Columns	Format	Name	Description
02	A1	ATYPE	Letter "U" must be coded for ATYPE if an upper bound is being imposed, letter "L" for a lower bound. An equality constraint uses either letter.
04-07	A4	ANAME	The name of the first of a sequence of CVFs to which the bound applies must be coded for ANAME (by sequence it is meant the order in which the CVFs appear in the code)
09-12	A4	ARG	The name of the last CVF in the above (uninterrupted) sequence to which the bound applies must be coded for ARG
17-28	F12.5 or 3A4	AC1†	• The value of the bound for the above CVF must be coded for AC1 • If the bound corresponds to a previously defined parameter, then the parameter name must be coded for AC1 (right justified)

†See the footnote of Table 7.

TABLE 9

THE TOLERANCE CARD

Columns	Format	Name	Description
01-12	E12.6	XEP1	Ignored by the program
13-24	E12.6	XEP2	When minimizing the P-function for a given value of r (RHØ) the value of P must decrease by an amount exceeding XEP2 for each iteration after the first. If it does not, then the code prints out the message "apparently roundoff errors prevent a more accurate determination of the minimum of this subproblem," and it is assumed that a minimum has been found. (Usually we set XEP2 equal to 0.)

12 The second option card. This card is read in by the program MAIN. The description of this card is given in Table 10.

TABLE 10

THE SECOND OPTION CARD

Option	Column	Value	Meaning
1	7	1	Ignored in this program
2	14	1	The method for minimizing the unconstrained penalty function is to be the generalized Newton-Raphson method as modified to handle indefinite Hessian matrices. This method requires function values, first and second derivatives
		2	Same as 1, except that when an "orthogonal move" is made because of an indefinite Hessian matrix, $-\nabla P$ is added to the orthogonal move vector
		3	Steepest descent is used to minimize the P-function
		4	The method for minimizing the unconstrained penalty function is the rank one method as reported in the Fiacco-McCormick book [10]. This requires function values and first derivatives

13 End card. This is the usual end card which must follow any coded problem. It contains "//" at the first two columns.

A schematic depiction of a problem deck structure is shown in Figure 1.

4. Coded Examples and Output Illustrations

For illustration purposes the coded input and the resulting output for the sample problem 1 discussed in Section 2 is reviewed in sufficient detail. Although the functional relationships involved in this problem are too simple and the problem itself is too small to illustrate the

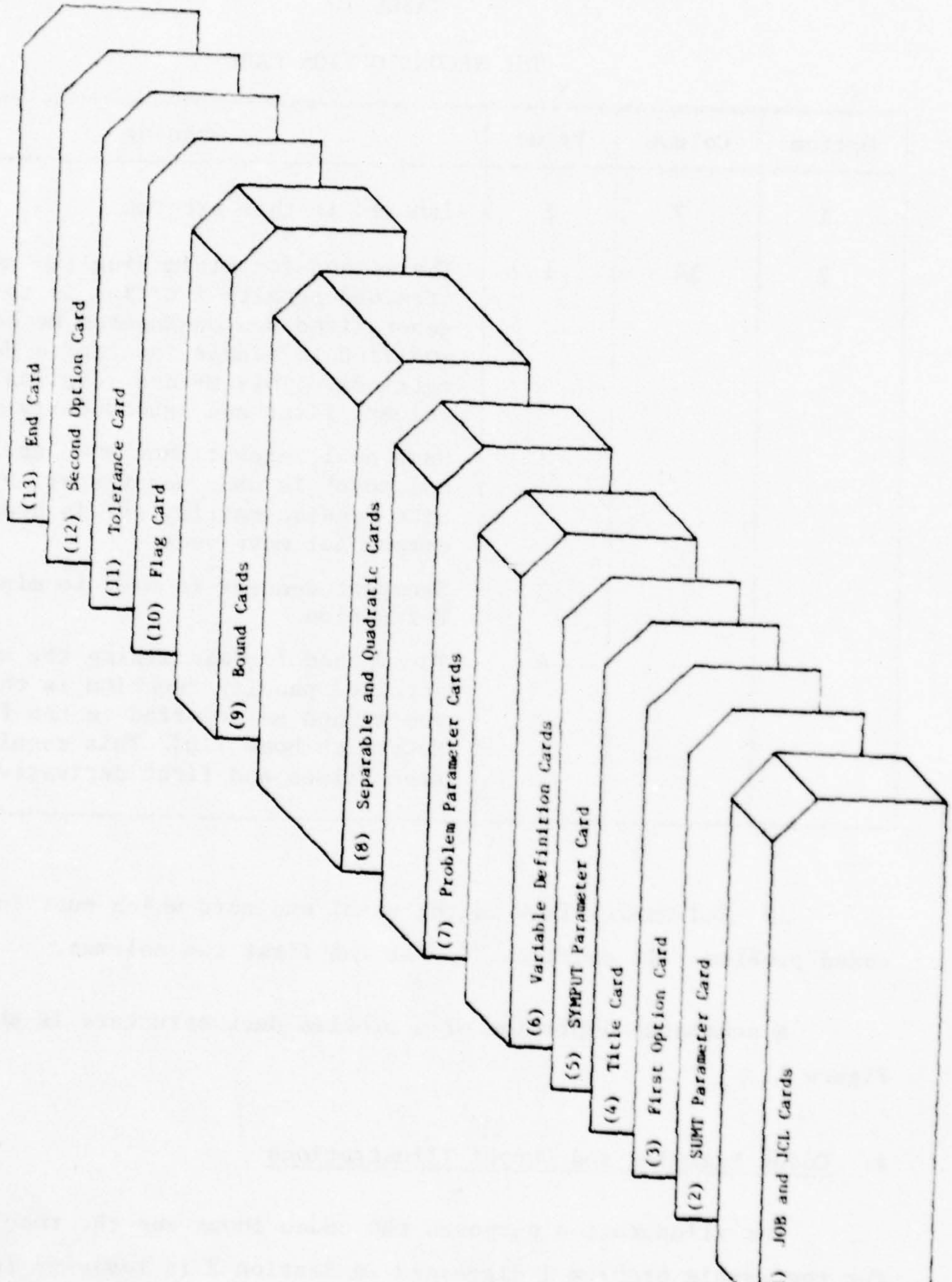


Figure 1.--Data deck structure.

merits of SYMBOLIC FACTORABLE SUMT, we have purposely chosen it so that users can readily follow and become familiar with the model with the least possible effort. For further elucidation the input/output for two more sample problems--Hopkin's problem from Reference [2] and the bulkhead design problem from Reference [5]--are included following the sample problem 1.

4.1 Sample problem 1 (a test problem).

4.1.1. Problem definition (restated for ease of reference):

$$\begin{aligned} \underset{x_1, x_2}{\text{minimize}} \quad & (x_1-10)^2 + (x_2-1)^2 \\ \text{subject to} \quad & C_5 \; x_2 - 5 \sin .5x_1 \leq C_1 \\ & x_2 - C_6 \; \ell n \quad x_1 \geq C_2 \\ & x_1 x_2 \geq C_3 \\ & x_1 \geq C_4 \, , \end{aligned}$$

where C_1 through C_6 are the parameters of the problem, and their respective values are

$$C_1 = C_2 = 0 \, , \quad C_3 = 3 \, , \quad C_4 = 1.5 \, , \quad C_5 = 1 \, , \quad C_6 = -1.5 \, .$$

Moreover, let us suppose that we intend to calculate the sensitivity and elasticity of the solution vector with respect to the parameters C_5 and C_6.

4.1.2. Symbolic factorable form. In SYMBOLIC FACTORABLE SUMT, use of canned routines to evaluate functions of one variable in a single iteration enables us to code the problem in the following concise form:

$$\min_{X \in E^6} X^6$$

symbolic factorable step

where $X^1 = x_1$ *(a)*

$X^2 = x_2$ *(b)*

$X^3 = C_5 X^2 - 5 \sin .5X^1$ *(c)*

$X^4 = X^2 - C_6 \ln X^1$ *(d)*

$X^5 = X^1 \cdot X^2$ *(e)*

$X^6 = (X^1 - 10)^2 + (X^2 - 1)^2$ *(f)*

subject to $X^3 \leq C_1$ *(g)*

$X^4 \geq C_2$ *(h)*

$X^5 \geq C_3$ *(i)*

$X^1 \geq C_4$. *(j)*

4.1.3. The keypunch sheet code. The keypunch sheets used to code the above problem in symbolic factorable language is shown in Figure 2. Table 11 shows the correspondence between the steps cited in 4.1.2 and the related symbolic code. Figure 3 depicts an annotated computer listing of the symbolic code for problem 1. The categories of the code are numbered and specified in the left margin of this figure in the same sequence as was illustrated in Section 3.

The symbolic names used in the code are directly taken from Table 12 at the end of the present section.

4.1.4. Output annotations. The computer output for the sample example 1 is depicted in Appendix 1, page 50. Various segments of this output are numbered in circles. Corresponding explanations are as follows:

1 Information contained in the title card

2 Printout of SYMPUT parameter card (the reader may skip reading steps 3-8).

```
//AAAAA JØB (5660,R),ABG
// EXEC FØRG6,DSN='ØR5660.ABG',PRØG=MAIN
//SYSLIB DD
//        DD
//        DD DSN=GWU.FØRTLIB,DISP=SHR
//GØ.FT05F001 DD *
    0.00001        100.0     0.00001         10.0        500.   4   2   0      01
     3      2      1      1      1      1      1      2      1      1          02
          EXAMPLE-1 : SØLUTIØN AND SENSITIVITY ANALYSIS                        03
   2    6    4    0    1    0    0    0    6                                  04
 V   X1                2.0                                                     05
 V   X2                2.0                                                     06
 P   C5                1.0                                                     07
 P   C6               -1.5                                                     08
 D   C1                0.0                                                     09
 D   C2                0.0                                                     10
 D   C3                3.0                                                     11
 D   C4                1.5                                                     12
 S   X3    X1 SIN      0.5         -5.0                                        13
 S   X3    X2 LIN       C5                                                     14
 S   X4    X2 LIN      1.0                                                     15
```

Figure 2.--Data cards for sample problem 1.

```
S   X4   X1 LNG        1.0            C6
Q   X5   X1 LIN        1.0
Q   X5   X2 LIN        1.0
S   X6   X1 QDR        1.0     -10.0
S   X6   X2 QDR        1.0      -1.0
U   X3   X3            C1
L   X4   X4            C2
L   X5   X5            C3
L   X1   X1            C4
5
     .001        .001
     1        1
//
```

Figure 2.--continued.

```
//AAAAAA JOB (5660,R).ABG
// EXEC FORG6,DSN='OR5660.ABG'.PROG=MAIN
//SYSLIB DD
//       DD
//       DD DSN=GWU.FORTLIB,DISP=SHR
//GO.FT05F001 DD *
    0.00001      100.0    0.00001        10.0      500.   4   2   0
     3       2      1      1      1      1      1      2      1      1
      EXAMPLE-1 : SOLUTION AND SENSITIVITY ANALYSTS
  2    6    4    0    1    0    0    0    6
V    X1                    2.0
V    X2                    2.0
P    C5                    1.0
P    C6                   -1.5
D    C1                    0.0
D    C2                    0.0
D    C3                    3.0
D    C4                    1.5
S    X3    X1 SIN          0.5          -5.0
S    X3    X2 LIN           C5
S    X4    X2 LIN          1.0
S    X4    X1 LNG          1.0           C6
Q    X5    X1 LIN          1.0
Q    X5    X2 LIN          1.0
S    X6    X1 QDR          1.0          10.0
S    X6    X2 QDR          1.0           1.0
U    X3    X3               C1
L    X4    X4               C2
L    X5    X5               C3
L    X1    X1               C4
5
   .001      .001
    1      1
/
```

1. JOB and JCL cards
2. SUMT parameter card
3. First option card
4. Title card
5. SYMPUT parameter card
6. Variable definition cards
7. Problem parameter cards (sensitivity calculations to be carried on parameters C5 and C6)
8. Separable and quadratic cards
9. Bound cards
10. Flag card
11. Tolerance card
12. Second option card
13. End card

Figure 3.--Computer listing of the sample problem 1 code.

TABLE 11

THE CORRESPONDENCE BETWEEN FUNCTIONS OF SAMPLE PROBLEM 1 IN SYMBOLIC FACTORABLE FORM AND ITS SYMBOLIC FACTORABLE CODE

Symbolic Factorable Step	Statement Serial Number in Keypunch Sheet (Figure 2)
(a)	05
(b)	06
(c)	13 and 14
(d)	15 and 16
(e)	17 and 18
(f)	19 and 20
(g)	21
(h)	22
(i)	23
(j)	24

Segments 3-8 pertain to the categorization of the coded symbolic input by the subroutine SYMPUT which is necessary for synthesis of the original problem from the symbolic code. In order to make these printouts self-explanatory, the reader should notice the following points, which are related to the programming logic of the subroutine SYMPUT:

(i) Separable terms are labeled with a type number "3" and and quadratic terms with a type number "4."

(ii) CVFs are sequentially labeled from 1 to NN (total number of CVFs) in the order of their appearance in the code.

(iii) All the problem parameters and constant coefficients are sequentially labeled by natural sequence of numbers in the order of their appearance in the code and stored in array CC1. Constants having the same numerical value are labeled with the same number. Moreover, in this array the problem parameters are separated from problem constant coefficients with numbers zero and one.

(iv) Constraints of the problem are sequentially labeled by natural sequence of numbers in the order of their appearance in the code.

(v) Each symbolic function, as discussed in Section 1, is associated with a unique number called a transformation code, which serves as a link to juxtapose the symbolic function name with its mathematical code.

With these points in mind we proceed to review briefly segments 3-8 of the output.

3 Modified printout of the symbolic code of the problem (excluding the bounds). For explanation of columns 2, (5,7), and 8 , see comments (i), (ii), and (v), respectively, above.

4 Manipulated printout of problem parameters and constants [see comment (iii)]

5 Bounds on problem constraints. For explanation see comments (iv) and (iii), respectively.

6 Number of times that a concomitant variable function (CVF) is defined in terms of a separable one-variable function or quadratic function (cumulative). For example, concomitant variable function X^4 is defined as follows:

$$X^4 = X^2 - C_2 \ell n X^1 ,$$

which has $5-3 = 2$ separable terms and $1-1 = 0$ quadratic terms.

7 Table of separable terms formed by subroutine SYMPUT (self-explanatory).

8 Table of quadratic terms set up by subroutine SYMPUT (self-explanatory).

9 Printout of parameter card.

10 Printout of first option card.

11 Printout of tolerance card.

12 Printout of second option card.

13 Printout of initial vector and corresponding problem function values.

14 Printout of feasible starting vector and corresponding problem function values (initial vector need not be feasible).

15 Objective function value, solution vector, constraint values, and Lagrange multipliers evaluated at solution point of subproblem 1 (at $r = 100$) . The following information is also evaluated at this solution point:

DOTT	: inverse of Hessian matrix of penalty function pre- and post-multiplied by penalty function gradient
MAGNITUDE:	norm of penalty function gradient
P	: value of the penalty function
G	: dual value
RSIGMA	: residual term related to inequality constraints in penalty function
H	: residual term related to equality constraint in penalty function.

16 Printout of the problem parameters on which sensitivity information is desired.

17 Zero*th* order sensitivity of the first component of the solution vector of subproblem 1 (i.e., x_1) and corresponding elasticities with respect to the problem parameters (repeated for all components of solution vector).

18 Solution to the second subproblem (at $r = 10$) and first order estimates of solution values.

19 First order sensitivity of the first component of solution vector of subproblem 2 and corresponding elasticities with respect to the problem parameters.

20 Solution to the third subproblem (at $r = 1$) and first and second order estimates of solution values.

21 Solution to the fourth subproblem (at $r = .1$) .

22 Solution to the fifth subproblem (at $r = .01$) .

23 Solution to the sixth subproblem (at $r = .001$) .

24 Solution to the seventh subproblem (at $r = .0001$) .

25 Solution to the eighth (final) subproblem (at $r = .00001$) .

Illustration of a few useful optional outputs seems beneficial at this stage.

- ITEST=0 rather than 1 (in SYMPUT parameter card) would suppress output segments numbered from 15 onwards, and would instead yield output segments A, B, C, and D, as labeled on page 64 of Appendix 1, and which have the following descriptions (all evaluated at the initial feasible point):

  A : the values of CVFs

  B : partial derivatives of the new CVFs with respect to the problem variables

  C : diagonal entries of the Hessian matrix of the new CVFs

  D :

- NSWIT=1 (in SYMPUT parameter card) would suppress the output segments 3-8.

- ITES=2 (in SYMPUT parameter card) would yield output segments E, F, G, H, I, J, and K after solution of each subproblem, which are defined as follows (all evaluated at the corresponding subproblem solution point):

  E : the entries of vector σ (see Table 4)

  F : the entries of vector α [see Equation (38), Appendix 2]

  G : the values of CVFs

  H : partials of the new CVFs with respect to the original problem variables premultiplied by vector α

  I : the joint partial derivatives of the new CVFs with respect to the original problem variables and sensitivity parameters premultiplied by vector α

  J : the partials of the new CVFs with respect to the sensitivity parameters

  K : the negative values of the zero*th* order sensitivity of the original problem variables with respect to the sensitivity parameters.

- NT8=1 (in first option card) would suppress the sensitivity and elasticity analysis calculations.

4.2 Sample problem 2 (Hopkins problem).

This problem has been solved in Reference [2] using FACTORABLE SUMT (earlier version of the SYMBOLIC FACTORABLE SUMT). Here it is chosen as the second sample problem for solution and sensitivity analysis by the SYMBOLIC FACTORABLE SUMT. For brevity, unlike the first sample problem, illustration of this sample problem is limited to the algebraic definition of the problem, computer listing of its code, and selected pages of the computer solution output.

4.2.1. Problem definition.

$$\text{minimize} \left(-100x_1^{.5} - 2x_2 - 25x_3^{.5} - 50x_4^{.5} - 10x_5 + 100(x_6-3)^2 - 30x_1^{.5}x_7^{.5} + 100x_8 - 10x_1/x_3 - 20x_1/x_4\right)$$

$$\text{subject to } g_1 : A_1x_1 + A_2x_2 + A_3x_3 + A_4x_4 + A_5x_3x_6 + A_6x_4x_6 - A_7x_4x_5x_6 + A_8x_1x_7 - A_9x_1x_8 \leq 25{,}000$$

$$g_2 : 12.25x_1 + x_3x_6 + .7x_4x_6 - .2x_4x_5x_6 \geq 34{,}000$$

$$g_3 : \quad x_5 \leq .5$$

$$g_4 : \quad x_8 \leq .3$$

$$g_5\text{-}g_{14} : \quad x_j \geq 0;\ j=1,8 ,$$

where the coefficients of the first constraint, i.e., A_1-A_9 , are chosen as problem parameters on which sensitivity and elasticity analysis is performed. The values of these parameters are:

$A_1 = 24$, $A_2 = 12$, $A_3 = .6$, $A_4 = .4$, $A_5 = .1$

$A_6 = .15$, $A_7 = -.15$, $A_8 = 11$, and $A_9 = 24$.

TABLE 12

LIST OF SINGLE VARIABLE FUNCTIONS USED BY SYMBOLIC FACTORABLE SUMT

Symbolic Name	Description	Algebraic Form	Default Options
CØN	Constant	C1	None
IDN	Identity	Z	None
LIN	Linear	C2 + C1*Z	C2 = 0
EXP	Exponential	C2* EXP(C1*Z)	C2 = 1, C1 = 1
EXC	--	$C2*\ (C1)^{Z}$	C2 = 1
PØW	Power	$C2*\ (Z)^{C1}$	C2 = 1
PØL	Polynomial	$C2*(Z)^{C1}$ (C1 integer)	C2 = 1
LNG	Natural Log	C2* LN(Z)	C2 = 1 [no C1]
LØG	Other Log	$C2*\ LØG_{C1}(Z)$	C2 = 1
SIN	Sine	C2* SIN(C1*Z)	C2 = 1, C1 = 1†
CØS	Cosine	C2* CØS(C1*Z)	C2 = 1, C1 = 1†
TAN	Tangent	C2* TAN(C1*Z)	C2 = 1, C1 = 1†
GAM	Gamma Function	$C2*\ \Gamma(C_1*Z)$	C2 = 1
NØR	Cumulative Normal	$C2*\ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{Z} EXP(\frac{-t^2}{2})dt$	C2 = 1
HEV	Heaviside§	C2* H(Z-C1)	C2 = 1

TABLE 12--*continued*

Symbolic Name	Description	Algebraic Form	Default Options
MIN	Minimum	C2* MIN[Z,C1]	C2 = 1
MAX	Maximum	C2* MAX[Z,C1]	C2 = 1
ARS	Arc Sine	C2* ARCSIN[C1*Z]	C2 = 1
ART	Arc Tangent	C2* ARCTAN[C1*Z]	C2 = 1
QDR	Quadratic	$[C1* Z-C2]^2$	None
ESQ	Exponential Square	$C2* EXP[C1*Z^2]$	C2 = 1
ENT	Entropy Function	C1* Z[C2 + LN(C1*Z)]	None
LGS	Logistic Curve	$[1 + C2* EXP(-C1*Z)]^{-1}$	C2 = 1
EXM	Decay Function	C2*[1 - EXP(-C1*Z)]	C2 = 1
LLN	Log of Linear Function	LN(-C1*Z + C2)	None
DIV	Division	C1/Z + C2	C1 = 1

†Default option C1 = 1 can be used only if the option C2 = 1 is also being exploited, i.e., C2 is not coded.

§Heaviside function is a step function defined as follows:

$$C2*H(Z-C1) = \begin{cases} 0 , & \text{if } Z < C1 \\ C2 , & \text{if } Z \geq C1 . \end{cases}$$

4.2.2. Computer listing of the symbolic code. (See Figure 4.)

4.2.3. Selected pages from computer output. (See Appendix 1, page 69.)

4.3 Sample problem 3 (corrugated transverse bulkhead design).

This problem, which has been taken from Reference [5], involves more complicated functional relationships than the two previous sample problems. It has been solved by many authors [6], [5], [7], [8]. A comparison of the time, cost, and effort spent to code and solve this problem in [5] and [8]--where the SUMT-Version 4 model was used--with those involved in the present study, shows the enormous advantage of the SYMBOLIC FACTORABLE SUMT model over the SUMT-Version 4 model. Table 13 accounts for this claim.

TABLE 13

SUMT-VERSION 4 VERSUS SYMBOLIC FACTORABLE SUMT ON SOLUTION OF OPTIMAL CORRUGATED TRANSVERSE BULKHEAD DESIGN PROBLEM

Comparison Criteria	SUMT-Version 4 [8]	SYMBOLIC SUMT [present study]
Number of statements used to code the problem	300^+	100
Approximate time spent formulating and coding the problem, hrs	25^+	4
CPU time spent solving the problem/run, seconds	61.63	27.64
Cost incurred in solving the problem/run, dollars	5.08^+	2.73

A comparison of the entries in the above table reveals how SYMBOLIC FACTORABLE SUMT facilitates the coding of problems and brings about savings in time and costs involved in the solution of nonlinear problems. As mentioned in Section 1, the primary reason for such reductions, especially in the coding of the problem, stems from the fact that SUMT-Version

```
//AAAAAA JOB (5660,R,3,3),ABG
// EXEC FORG6,DSN='OR5660.ABG',PROG=MAIN
//SYSLIB DD
//        DD
//        DD DSN=GWU.FORTLIB,DISP=SHR
//GO.FT05F001 DD *
     0.00001         100.0        .00001         10.0          300.  12   8   0
       3        2       2      1       1       1       1      2       1       1
EXAMPLE-2 : SOLUTION AND SENSITIVITY ANALYSIS OF HOPKINS PROBLEM
     8   14   12     0     1     0     0     0     9
  V    X1                 180.00
  V    X2                 740.00
  V    X3                2680.00
  V    X4               14650.00
  V    X5                    .40
  V    X6                   2.30
  V    X7                    .60
  V    X8                    .20
  P    A1                  24.00
  P    A2                  12.00
  P    A3                  0.600
  P    A4                   0.40
  P    A5                   0.10
  P    A6                   0.15
  P    A7                  -0.15
  P    A8                  11.00
  P    A9                 -24.00
  Q    X9    X4 IDN
  Q    X9    X6 IDN
  Q   X10    X3 IDN
  Q   X10    X6 IDN
  S   X11    X1 LIN           A1
  S   X11    X2 LIN           A2
  S   X11    X3 LIN           A3
  S   X11    X4 LIN           A4
  S   X11   X10 LIN           A5
  S   X11    X9 LIN           A6
  Q   X11    X9 LIN           A7
  Q   X11    X5 IDN
  Q   X11    X1 LIN           A8
  Q   X11    X7 IDN
  Q   X11    X1 LIN           A9
  Q   X11    X8 IDN
  S   X12    X1 LIN        12.25
  S   X12   X10 IDN
  S   X12    X9 LIN         0.70
  Q   X12    X9 LIN        -0.20
  Q   X12    X5 IDN
```

Figure 4.--Computer listing of sample problem 2.

```
S   X13   X6  QDR       1.00
S   X14   X1  POW       0.50          3.
S   X14   X2  LIN      -2.00       -100.
S   X14   X3  POW       0.50
S   X14   X4  POW       0.50        -25.
S   X14   X5  LIN     -10.00        -50.
S   X14   X13 LIN     100.00
Q   X14   X1  POW       0.50
Q   X14   X7  POW       0.50        -30.
S   X14   X8  LIN     100.00
Q   X14   X1  LIN     -10.00
Q   X14   X3  POW      -1.
Q   X14   X1  LIN     -20.00
Q   X14   X4  POW      -1.
U   X11   X11       25000.00
L   X12   X12       34000.00
U   X5    X5           .50
U   X8    X8           .30
L   X1    X8          0.00
5
    0.001       .001
     1         1
//
```

Figure 4.--continued

4 requires explicit derivations and coding of the problem function gradients and Hessians (user's subroutine), while in SYMBOLIC FACTORABLE SUMT these requirements are internally calculated and so are relaxed. The derivation and coding of these entries is sometimes extremely tedious and time consuming, as is the case for the present problem, and is prone to mathematical errors whose debugging is another factor in the user's time consumption. It was this very fact that motivated the development of SYMBOLIC FACTORABLE SUMT, and the present example clearly demonstrates that this goal has been very well achieved.

It is not the intention of the present work, or within its scope, to discuss the engineering aspects of this optimal design problem and the significance of its constraints. Readers interested in such aspects should refer to [5]. Instead, similar to the sample problem 2, its discussion is limited to the algebraic definition of the problem, a computer listing of its code, and selected pages from computer solution output.

4.3.1. Problem definition. Following is the algebraic definition of the problem, borrowed from [5]. For sensitivity analysis considerations, the coefficients of K_2 and $t_{\bullet}^{min}$ in constraints 8-16 have been normalized to unity in the code of the problem (i.e., constraints 8, 11, and 14 have been scaled by -1, and constraints 9, 10, 11, 12, 13, 15, and 16 by $-\frac{1}{10}$). The problem is

$$\text{minimize} \quad W = \Gamma B(x_1+x_2)(\ell_t x_4+\ell_m x_5+\ell_b x_6)[x_1 + (x_2^2-x_3^2)^{\frac{1}{2}}]^{-1}$$

subject to *constraint number*

$$x_2 - x_3 \geq 0 \,. \qquad (1)$$

$$\frac{1}{6}x_2x_3x_4 + \frac{e}{2}x_1x_3x_4 - K_1h_t\ell_t^2[x_1 + (x_2^2-x_3^2)^{\frac{1}{2}}] \geq 0 \,, \qquad (2)$$

$$\frac{1}{6}x_2x_3x_5 + \frac{e}{2}x_1x_3x_5 - K_1h_m\ell_m^2[x_1 + (x_2^2-x_3^2)^{\frac{1}{2}}] \geq 0 \,, \qquad (3)$$

$$\frac{1}{6}x_2x_3x_6 + \frac{e}{2}x_1x_3x_6 - K_1h_b\ell_b^2[x_1 + (x_2^2-x_3^2)^{\frac{1}{2}}] \geq 0 \,. \qquad (4)$$

$$\frac{1}{12}\, x_2 x_3^2 x_4 + \frac{e}{4}\, x_1 x_3^2 x_4 - 2.2(K_1 h_t \ell_t^2)^{4/3}[x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}]^{4/3} \geq 0 \,, \tag{5}$$

$$\frac{1}{12}\, x_2 x_3^2 x_5 + \frac{e}{4}\, x_1 x_3^2 x_5 - 2.2(K_1 h_m \ell_m^2)^{4/3}[x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}]^{4/3} \geq 0 \,, \tag{6}$$

$$\frac{1}{12}\, x_2 x_3^2 x_6 + \frac{e}{4}\, x_1 x_3^2 x_5 - 2.2(K_1 h_b \ell_b^2)^{4/3}[x_1 + (x_2^2 - x_3^2)^{\frac{1}{2}}]^{4/3} \geq 0 \,. \tag{7}$$

$$x_4 - t_t^{min} \geq 0 \,, \tag{8}$$

$$10x_4 - [3.9 \cdot 1.05(.01h_{1t})^{\frac{1}{2}}(.01x_1) + 10K_2] \geq 0 \,, \tag{9}$$

$$10x_4 - [3.9 \cdot 1.05(.01h_{1t})^{\frac{1}{2}}(.01x_2) + 10K_2] \geq 0 \,, \tag{10}$$

$$x_5 - t_m^{min} \geq 0 \,, \tag{11}$$

$$10x_5 - [3.9 \cdot 1.05(.01h_{1m})^{\frac{1}{2}}(.01x_1) + 10K_2] \geq 0 \,, \tag{12}$$

$$10x_5 - [3.9 \cdot 1.05(.01h_{1m})^{\frac{1}{2}}(.01x_2) + 10K_2] \geq 0 \,, \tag{13}$$

$$x_6 - t_b^{min} \geq 0 \,, \tag{14}$$

$$10x_6 - [3.9 \cdot 1.05(.01h_{1b})^{\frac{1}{2}}(.01x_1) + 10K_2] \geq 0 \,, \tag{15}$$

$$10x_6 - [3.9 \cdot 1.05(.01h_{1b})^{\frac{1}{2}}(.01x_2) + 10K_2] \geq 0 \,. \tag{16}$$

$$x_i \geq 0; \; i=1,2, \text{ and } 3 \,.$$

The listing of the problem parameters and their values is in Table 14.

4.3.2. Computer listing of the symbolic code. (See Figure 5.)

4.3.3. Selected pages from computer output. (See Appendix 1, page 78.)

TABLE 14

PROBLEM DESIGN PARAMETERS
(SAMPLE PROBLEM 3)

Input Data		
$B =$	476	cm
$\Gamma =$	7.85	ton/cm
* $\ell_t =$	495	cm
* $\ell_m =$	385	cm
* $\ell_b =$	315	cm
$h_t =$	498	cm
$h_m =$	938	cm
$h_b =$	1288	cm
$h_{1t} =$	745	cm
$h_{1m} =$	1130	cm
$h_{1b} =$	1445	cm
* $t_t^{min} =$	1.05	cm
* $t_m^{min} =$	1.05	cm
* $t_b^{min} =$	1.05	cm
$e =$	0.8	cm
$k_1 =$	6.94×10^{-8}	cm^{-1}
* $k_2 =$	0.15	cm
$h_a =$	250	cm

*Parameters marked with an asterisk are selected for sensitivity analysis.

```
//AAAAAA JOB (5660,R,4,4),ABG
// EXEC FORG6,DSN='OR5660.ABG',PROG=MAIN
//SYSLIB DD
//       DD
//       DD DSN=GWU.FORTLIB,DISP=SHR
//GO.FT05F001 DD *
  0.00001              100.    .000001            4.0          800.  19    6    0
     3        2       2        1       1       1      1       2      1       1
EXAMPLE-3 : SOLUTION AND SESITIVITY ANALYSIS OF CORRUGATED BULKHEAD
    6   45    19     0        1                        15
  V    X1                   45.8
  V    X2                   43.2
  V    X3                   30.5
  V    X4                    1.2
  V    X5                    1.2
  V    X6                    1.3
  P    LT                   495.
  P    LM                   385.
  P    LB                   315.
  P   TTM                   1.05
  P   TMM                   1.05
  P   TBM                   1.05
  P    K2                   0.15
  D     E                     .8
  D    GB                 3736.6
  D   KHT             0.00003456
  D   KHM             0.00006510
  D   KHB             0.00008938
  D    CT                .011177
  D    CM                .013765
  D    CB                .015566
  S    Z1    X2 POW           2.
  S    Z1    X3 POW           2.            -1.
  S    Z2    X1 IDN
  S    Z2    Z1 POW           .5
  Q    Z3    X3 LIN           .5
  Q    Z3    X4 IDN
  Q    Z4    X3 LIN           .5
  Q    Z4    X5 IDN
  Q    Z5    X3 LIN           .5
  Q    Z5    X6 IDN
  S    Z6    X2 LIN   0.33333333
  S    Z6    X1 LIN            E
  Q    Z7    Z3 IDN
  Q    Z7    Z6 IDN
  Q    Z8    Z4 IDN
  Q    Z8    Z6 IDN
  Q    Z9    Z5 IDN
  Q    Z9    Z6 IDN
  S   Z10    Z2 LIN          KHT
  S   Z11    Z2 LIN          KHM
  S   Z12    Z2 LIN          KHB
  S   Z13   Z10 LIN           LT
  S   Z14   Z11 LIN           LM
  S   Z15   Z12 LIN           LB
```

Figure 5.--Computer listing of sample problem 3.

```
S   Z16  Z13 LIN          LT
S   Z17  Z14 LIN          LM
S   Z18  Z15 LIN          LB
S    G1   X2 IDN
S    G1   X3 LIN         -1.
S    G2   Z7 IDN
S    G2  Z16 LIN         -1.
S    G3   Z8 IDN
S    G3  Z17 LIN         -1.
S    G4   Z9 IDN
S    G4  Z18 LIN         -1.
Q    G5   Z7 IDN
Q    G5   X3 LIN          .5
S    G5  Z16 POW  1.33333333         -2.2
Q    G6   Z8 IDN
Q    G6   X3 LIN          .5
S    G6  Z17 POW  1.33333333         -2.2
Q    G7   Z9 IDN
Q    G7   X3 LIN          .5
S    G7  Z18 POW  1.33333333         -2.2
S    G8   X4 LIN         -1.          TTM
S    G9   X4 LIN         -1.
S    G9   X1 LIN          CT           K2
S   G10   X4 LIN         -1.
S   G10   X2 LIN          CT           K2
S   G11   X5 LIN         -1.          TMM
S   G12   X5 LIN         -1.
S   G12   X1 LIN          CM           K2
S   G13   X5 LIN         -1.
S   G13   X2 LIN          CM           K2
S   G14   X6 LIN         -1.          TBM
S   G15   X6 LIN         -1.
S   G15   X1 LIN          CB           K2
S   G16   X6 LIN         -1.
S   G16   X2 LIN          CB           K2
S   Z19   X1 IDN
S   Z19   X2 IDN
S   Z20  Z19 LIN          GB
S   Z21   X4 LIN          LT
S   Z21   X5 LIN          LM
S   Z21   X6 LIN          LB
Q   Z22  Z20 IDN
Q   Z22  Z21 IDN
Q   OBJ  Z22 IDN
Q   OBJ   Z2 POW         -1.
L    X1   X3              0.
L    G1   G7              0.
U    G8  G16              0.
5
         .001        .001
     1          1
//
```

Figure 5.--continued

5. General Description of the New Subroutines

The following is a list of the names of the subroutines that comprise the SYMBOLIC FACTORABLE SUMT model in alphabetical order:

BODY	OUTPUT
CHCKER	PARSEN*
CONVRG	PEVALU
DLPEN*	PUNCH
ESTIM	REJECT
EVALU	RESTNT**
EXDELP*	RHOCOM
FEAS	SECORD
FINAL	SENS*
GRAD	STORE
GRAD1**	SYMPUT*
GVALU*	TCHECK
INVERS	TIMEC
MAIN	VALU*
MATRIX**	XMOVE
OPT	

Subroutines marked with a single asterisk (*) are the new subroutines that have been added to the SUMT-Version 4 model in its various developmental stages to result in SYMBOLIC SENSITIVITY SUMT model. Subroutines marked with a double asterisk (**) correspond to the user's subroutine in SUMT-Version 4.

In order to provide problem function values, their gradients and Hessian matrices, users had to code the latter subroutines in SUMT-Version 4, while here, in conjunction with the new subroutines, they are modified to serve the same purpose internally. The remainder of the subroutines virtually serve the same purposes as in SUMT-Version 4.

For brevity, a general description of the new subroutines is provided here. Readers interested in general descriptions of the remaining subroutines must refer to [1].

The new subroutines fall into two categories,

(i) those developed to relax the user from coding old users' subroutines--these subroutines are SYMPUT, VALU, and GVALU; and

(ii) those developed to perform sensitivity and elasticity calculations--these subroutines are PARSEN, SENS, EXDELP, and DLPEN.

Following is a brief and general description of the role of each of these subroutines.

SYMPUT

Reads in and stores in arrays the coded problem in symbolic factorable format such that problem functions can be synthesized with the aid of these arrays. The title, SYMPUT parameter, and flag cards are also read in by this subroutine. It prints out a modified listing of the coded problem if desired. As described in Section 3 (Tables 7 and 8), this subroutine selects F or A format internally, while reading the coded problem, for numeric and alphanumeric inputs, respectively. This is made possible with the aid of subroutine REREAD, which is a special purpose George Washington University FORTRAN Library routine. Following is a brief description of this subroutine.

"Subroutine Name: REREAD - permits reformating of a record.

Description: REREAD patches the FORTRAN I/O routine so that upon execution it examines every read for a data set reference number (device code) of 99. If 99 is not found normal reading takes place. If 99 is found reading does not take place and the record read by the last normal read is formated according to the new format. The call needs to be issued only once for the program.

Usage: CALL REREAD

Example:

```
      DIMENSION A(10),B(20)
      CALL REREAD
      ..........
      ..........
      READ (5,100) A   This will read 10 variables
                       on a card.
      READ (99,101) B  This will reformat the card
                       into 20A1.
      ..........
      ..........
100   FORMAT(10F2.0)
101   FORMAT(20A1)."
```

VALU

Evaluates the functional values, first and second derivatives of single variable functions as required by GVALU. These results are transferred to subroutine GVALU via common blocks to reconstruct the problem function values, gradients, and Hessian matrices.

GVALU

Using the information stored in subroutine SYMPUT regarding the coded problem, via common blocks, GVALU reconstructs the problem function values, gradients, and Hessian matrices with the aid of subroutine VALU.

PARSEN

Acts as a coordinating routine to conduct sensitivity analysis in the solution vector of each subproblem. It also calculates the first order sensitivity information, elasticities of solution vector with respect to the problem parameters, and prints them out.

SENS

Evaluates the functional values, first and second derivatives of single variable functions, partial derivatives of these functions with respect to function parameters (C1 and C2), and joint partials with respect to parameters and function variables as required by subroutine EXDELP. This information is transferred to subroutine EXDELP for reconstruction of functional values, gradients, Hessian matrices of the problem functions, and related partial derivatives and cross partial matrices required for sensitivity analysis.

EXDELP

Using the information stored in the subroutine SYMPUT about the coded problem, and utilizing the information calculated in SENS, via common blocks, reconstructs the functional values, gradients, Hessian matrices of the problem functions, their partials with respect to the problem parameters, and joint partials with respect to the problem variables and parameters.

DLPEN

Using the information calculated in EXDELP via common blocks, uses the sensitivity formulas given in Appendix 2, to calculate the zero*th* order sensitivity of the solution vector with respect to the problem parameters.

An important point worth noting here is that the subroutines SYMPUT, VALU, GVALU, SENS, and EXDELP are quite general subroutines that can be interfaced with any nonlinear programming algorithm which uses the first or the first and second derivative methods.

Subroutines comprising SYMBOLIC FACTORABLE SUMT are currently dimensioned to handle problems of the following maximum scale:

- total number of CVFs $\leq$ 100
- number of original variables $\leq$ 30
- total number of parameters $\leq$ 75
- number of sensitivity parameters $\leq$ 15
- total number of original problem constraints $\leq$ 200 .

However, if computer capacity permits, they can readily be redimensioned to solve problems of higher dimensions. All of these subroutines are separately filed in the TED facility [9] at The George Washington University under their corresponding names. They may be accessed from the above source using account number 55599 and the proper password.

REFERENCES

[1] MYLANDER, W. C., R. L. HOLMES, and G. P. McCORMICK (1971). A guide to SUMT-Version 4: The computer program implementing the sequential unconstrained minimization technique for nonlinear programming. RAC Paper RAC-P-63, Research Analysis Corporation, McLean, Virginia.

[2] McCORMICK, G. P. (1974). A mini-manual for use of the SUMT computer program and the factorable programming language. Technical Report SOL 74-15, System Optimization Laboratory, Department of Operations Research, Stanford University, Stanford, California.

[3] deSILVA, A. H. and G. P. McCORMICK (1977). An interim manual for using the SUMT algorithm in conjunction with the symbolic factorable programming language. Draft paper.

[4] deSILVA, A. H. and G. P. McCORMICK (1978). Sensitivity analysis in nonlinear programming using factorable input. Technical Paper Serial T-365, Institute for Management Science and Engineering, The George Washington University.

[5] BRACKEN, J. and G. P. McCORMICK (1968). *Selected Applications of Nonlinear Programming*. John Wiley and Sons, Inc., New York.

[6] KAVLIE, D., J. KOWALIK, and J. MOE (1966). Structural optimization by means of nonlinear programming. Department of Ship Structures, Technical University of Norway, Trondheim.

[7] McCORMICK, G. P. (1975). Optimal design of a corrugated transverse bulkhead; an example of the use of the factorable programming language. Technical Paper Serial T-313, Institute for Management Science and Engineering, The George Washington University.

[8] FIACCO, A. V. and A. GHAEMI (1979). Sensitivity analysis of a nonlinear structural design problem. In preparation.

[9] TED FACILITY USER'S GUIDE, V2MO (1977). Center for Academic and Administrative Computing, George Washington University.

[10] FIACCO, A. V. and G. P. McCORMICK (1968). *Nonlinear Programming: Sequential Unconstrained Minimization Techniques*. John Wiley and Sons, Inc., New York.

APPENDIX 1

Annotated Computer Output for Solution of Sample Problem 1 (pages 50-68) and Selected Pages of Computer Solution Output for Sample Problems 2 (pages 69-77) and 3 (pages 78-84)

```
                              (1)     EXAMPLE-1 : SOLUTION AND SENSITIVITY ANALYSIS
  SYMPUT PARAMETER CARD
(2) N=  2   NN=  6   M=  4   MZ=  0   ITEST=  1   IPR=  0   ITES=  0   NSWIT=  0   IID=  6

    TYPE  TYPE  TYPE      VAR.  NUMBR.    ARG.     NUMBR.      FUNCT. NUMBR.  CONSTANT        CONSTANT
    NAME   #    DESCRIP.  NAME  ASSND.    VARIAB.               TYPE  ASSND.    C1              C2
                                          NAME

    V   ORIG. VARIABLE    X1    1    0.20000D 01
    V   ORIG. VARIABLE    X2    2    0.20000D 01

    P  SENSITIVITY PAR    C5                                            1    0.10000D 01

    P  SENSITIVITY PAR    C6                                            2   -0.15000D 01
    D  DATA DEFINITION    C1                                                0.0
    D  DATA DEFINITION    C2                                                0.0
(3) D  DATA DEFINITION    C3                                                0.300000000D 01
    D  DATA DEFINITION    C4                                                0.150000000D 01
    S     3   SEPARABL    X3    3         X1        1          SIN     20 0.500000000D 00  -.500000000D 01
    S     3   SEPARABL    X3    3         X2        2          LIN      3 0.100000000D 01  0.0
                                                                          C5
    S     3   SEPARABL    X4    4         X2        2          LIN      3 0.100000000D 01  0.0
    S     3   SEPARABL    X4    4         X1        1          LNG     15 0.100000000D 01  -.150000000D 01
                                                                                            C6
    Q     4   QUADRATC    X5    5         X1        1          LIN      3 0.100000000D 01  0.0
    Q     4   QUADRATC    X5    5         X2        2          LIN      3 0.100000000D 01  0.0
    S     3   SEPARABL    X6    6         X1        1          QDR     33 0.100000000D 01  0.100000000D 02
    S     3   SEPARABL    X6    6         X2        2          QDR     33 0.100000000D 01  0.100000000D 01

     ARRAY OF CONSTANTS CC1(.)
     *******************

       LOCATION     VALUE

            1      0.10000D 01
            2     -0.15000D 01
            3      0.0
(4)         4      0.0
            5      0.30000D 01
            6      0.15000D 01
            7      0.0
            8      0.10000D 01
            9      0.50000D 00
           10     -0.50000D 01
           11      0.10000D 02

     BOUNDS:NCONC(.,.)  : LOWER BOUND HAS MINUS SIGN

     CVF    CVF#  VALUE OF     LOCATION
                   BOUND       IN CC1(.)
(5)  1   X3     3  0.0             3
     2   X4     4  0.0            -4
     3   X5     5  0.3000000 01   -5
     4   X1     1  0.1500000 01   -6

     RELATIVE LOCATION:ITAB(.,.)
   CVF# SEPARABLE QUADRATIC
     3      1         1
(6)  4      3         1
     5      5         1
     6      5         2
     7      7         2
```

7

	ARG	ARG#	TRAN#	TRANSF	C1	LOC.IN CC1	C2	LOC.IN CC1
1	X1	1	20	SIN	0.5000D 00	9	-.5000 01	10
2	X2	2	3	LIN	0.1000D 01	1	0.0	7
3	X2	2	3	LIN	0.1000D 01	8	0.0	7
4	X1	1	15	LNG	0.1000D 01	8	-.1500 01	2
5	X1	1	33	QDR	0.1000D 01	8	0.1000 02	11
6	X2	2	33	QDR	0.1000D 01	8	0.1000 01	8

QUADRATIC DEFINITIONS : IUADTB(.,.) : EACH LINE REFERS TO A PAIR OF ARGUMENTS

8

	FIRST ARGUMENT									SECOND ARGUMENT							
	ARG	ARG#	TRAN#	TRANSF	C1	LOC.IN CC1	C2	LOC.IN CC1		ARG	ARG#	TRAN#	TRANSF	C1	LOC.IN	C2	LOC.IN
1	X1	1	3	LIN	0.1000D 01	8	0.0	7	*	X2	2	3	LIN	0.1000D 01	8	0.0	7

```
[illegible] ROUTINE-SUMT VERSION 4 08/10/71
(9)  N=  2      M=  4      MZ=  0
        MAX. TIME= 0.50000000 03    R= 0.10000000 03    RATIO= 0.10000000 02      EPSILON= 0.1000000D-04    THETA= 0.1000000D-04
(10) OPTIONS SELECTED
       3     2     1     1     1     1     1     2     1     1
(11) TOLERANCES
       0.1000000D-02        0.1000000D-02
(12) SECOND SET OF OPTIONS
       1     1
         F=  0.65000000 02      P=  0.0                G=  0.0                RSIGMA=  0.0              H=  0.0
        THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
     ---------------------------------------------------------------------
(13)  X1=  0.20000000D 01     X2=  0.20000000D 01
        THE CONSTRAINT VALUES
         0.22073550 01          0.96027920 00          0.10000000 01          0.50000000 00

     *****THE FEASIBLE STARTING POINT TO BE USED IS ...
         F=  0.65000000 02      P=  0.0                G=  0.0                RSIGMA=  0.0              H=  0.0
(14)    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
     ---------------------------------------------------------------------
      X1=  0.20000000D 01     X2=  0.20000000D 01
        THE CONSTRAINT VALUES
         0.22073550 01          0.96027920 00          0.10000000 01          0.50000000 00
     **********************************************
            POINT=   3        DOTT=  0.95411290-05      RHO=  0.10000000 03      MAGNITUDE=  0.31566160-01      PHASE= 2
         F=  0.46097530 02      P= -0.33441220 03      G= -0.35390250 03      RSIGMA= -0.38050970 03      H=  0.0
        THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
     ---------------------------------------------------------------------
      X1=  0.37505537D 01     X2=  0.36536666D 01
        THE CONSTRAINT VALUES
(15)     0.11163480 01          0.16708110 01          0.10703270 02          0.22505540 01

      LAGRANGE MULTIPLIERS
         F=  0.46097530 02      P= -0.33441220 03      G= -0.35390250 03      RSIGMA= -0.38050970 03      H=  0.0
        THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
     ---------------------------------------------------------------------
      X1=  0.37505537D 01     X2=  0.36536666D 01
        THE CONSTRAINT VALUES
         0.89577840 02          0.59851160 02          0.93429370 01          0.44433510 02

     SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
     *********************************************************

     PROBLEM PARAMETERS ARE
     ======================
(16)  C5          C6

     THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
     -----------------------------------------------------------------------------------------
     -0.465140 00  0.314990 00

     RESPECTIVE ELASTICITIES ARE :
```

```
 -0.12402D 00 -0.12598D 00

THE ESTIMATES OF ZERUTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.27637D 01 -0.48034D 00

RESPECTIVE ELASTICITIES ARE :
=============================

 -0.75642D 00  0.19720D 00
APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

*******************************************
      POINT=    6       DOTT=  0.1471231D-04       RHO=  0.1000000D 02       MAGNITUDE=  0.2037057D-01       PHASE= 2
(18)   F=  0.3680283D 02       P=  0.3288146D 01      G= -0.3197171D 01      RSIGMA= -0.3351468D 02         H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
--------------------------------------------------------------------
   X1=  0.43303384D 01     X2=  0.31581859D 01
     THE CONSTRAINT VALUES
      0.9843133D 00        0.9597174D 00        0.1067601D 02        0.2830338D 01

    1ST  ORDER ESTIMATES
       F=  0.3584189D 02       P=  0.4081041D 02      G=  0.3577009D 02      RSIGMA=  0.0               H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
--------------------------------------------------------------------
   X1=  0.43947589D 01     X2=  0.31031325D 01
     THE CONSTRAINT VALUES
      0.9470465D 00        0.8825135D 00        0.1063752D 02        0.2894759D 01

   LAGRANGE MULTIPLIERS
       F=  0.3584189D 02       P=  0.4081041D 02      G=  0.3577009D 02      RSIGMA=  0.0               H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
--------------------------------------------------------------------
   X1=  0.43947589D 01     X2=  0.31031325D 01
     THE CONSTRAINT VALUES
      0.1015937D 02        0.1041973D 02        0.9366792D 00        0.3533146D 01

SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
*****************************************************

PROBLEM PARAMETERS ARE
======================

  C5       C6

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.87601D 00  0.48925D 00

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------
(19)
-0.92166D 00  0.50861D 00

RESPECTIVE ELASTICITIES ARE :
=============================
```

```
THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-----------------------------------------------------------------------------------------------

-0.13339D 01 -0.85640D 00

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-----------------------------------------------------------------------------------------------

-0.11750D 01 -0.89818D 00

RESPECTIVE ELASTICITIES ARE :
=============================

 -0.37206D 00  0.42660D 00

********************************************
(20)    POINT=  10        DOTT=  0.5886959D-05        RHO=  0.1000000D 01        MAGNITUDE=  0.1815338D-01        PHASE= 2
       F=  0.2673064D 02      P=  0.2645202D 02      G=  0.2273064D 02      RSIGMA= -0.2786173D 00      H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
   X1=  0.50790812D 01      X2=  0.25859375D 01
     THE CONSTRAINT VALUES
      0.2457370D 00         0.1482419D 00         0.1013419D 02         0.3579081D 01

    2ND  ORDER ESTIMATES
       F=  0.2562837D 02      P=  0.2890675D 02      G=  0.2561151D 02      RSIGMA=  0.0      H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
   X1=  0.51700276D 01      X2=  0.25164879D 01
     THE CONSTRAINT VALUES
      0.1249347D 00         0.5217083D-01         0.1001031D 02         0.3670028D 01

    1ST  ORDER ESTIMATES
       F=  0.2572115D 02      P=  0.2902579D 02      G=  0.2561151D 02      RSIGMA=  0.0      H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
   X1=  0.51622749D 01      X2=  0.25223543D 01
     THE CONSTRAINT VALUES
      0.1355048D 00         0.6028828D-01         0.1002109D 02         0.3662275D 01

  LAGRANGE MULTIPLIERS
       F=  0.2572115D 02      P=  0.2902579D 02      G=  0.2561151D 02      RSIGMA=  0.0      H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
   X1=  0.51622749D 01      X2=  0.25223543D 01
     THE CONSTRAINT VALUES
      0.4069391D 01         0.6745731D 01         0.9867590D-01         0.2794013D 00

SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
*****************************************************

PROBLEM PARAMETERS ARE
======================

  C5        C6

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-----------------------------------------------------------------------------------------------
```

```
-0.101120 01  0.654100 00

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.102620 01  0.672420 00

RESPECTIVE ELASTICITIES ARE :
*****************************

 -0.202040 00 -0.198580 00

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.403270 00 -0.136720 01

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.299860 00 -0.142390 01

RESPECTIVE ELASTICITIES ARE :
*****************************

 -0.115960 00  0.825950 00
APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

*******************************************
       POINT=  13        DOTT=  0.6592370D-04     RHO=  0.1000000D 00      MAGNITUDE=  0.2027132D 00       PHASE= 2
21       F=  0.2503643D 02     P=  0.2544952D 02     G=  0.2463643D 02     RSIGMA=  0.4130901D 00     H=  0.0
      THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------
  X1=  0.52250954D 01    X2=  0.24955652D 01
    THE CONSTRAINT VALUES
     0.2798036D-01        0.1535560D-01        0.1003957D 02        0.3725095D 01


    2ND   ORDER ESTIMATES
       F=  0.2484312D 02     P=  0.2530088D 02     G=  0.2484818D 02     RSIGMA=  0.0                H=  0.0
      THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------
  X1=  0.52421176D 01    X2=  0.24851518D 01
    THE CONSTRAINT VALUES
     0.1565042D-02        0.6348279D-04        0.1002746D 02        0.3742118D 01


    1ST   ORDER ESTIMATES
       F=  0.2485182D 02     P=  0.2533813D 02     G=  0.2484818D 02     RSIGMA=  0.0                H=  0.0
      THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------
  X1=  0.52413192D 01    X2=  0.24855238D 01
    THE CONSTRAINT VALUES
     0.2924514D-02        0.6639904D-03        0.1002742D 02        0.3741319D 01


  LAGRANGE MULTIPLIERS
       F=  0.2485182D 02     P=  0.2533813D 02     G=  0.2484818D 02     RSIGMA=  0.0                H=  0.0
      THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------
  X1=  0.52413192D 01    X2=  0.24855238D 01
    THE CONSTRAINT VALUES
     0.3573936D 01        0.6512281D 01        0.9960593D-02        0.2684495D-01
```

```
SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
**********************************************************

PROBLEM PARAMETERS ARE
======================

  C5          C6


THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.963090D 00  0.654770D 00

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.957750D 00  0.654840D 00

RESPECTIVE ELASTICITIES ARE :
=============================

 -0.183300D 00 -0.187990D 00


THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.286170D 00 -0.141480D 01

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.273160D 00 -0.142010D 01

RESPECTIVE ELASTICITIES ARE :
=============================

 -0.109460D 00  0.853590D 00

********************************************
        POINT=  15      DOTT=   0.5655324D-06       RHO=  0.1000000D-01        MAGNITUDE=  0.6452718D-01        PHASE= 2
22    F=  0.2485718D 02       P=  0.2494437D 02      G=  0.2481718D 02      RSIGMA=  0.8718882D-01       H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------------
   X1=  0.52410000D 01     X2=  0.24863049D 01
     THE CONSTRAINT VALUES
      0.2835304D-02          0.1536448D-02          0.1003072D 02          0.3741000D 01


    2ND  ORDER ESTIMATES
      F=  0.2483716D 02       P=  0.2488370D 02      G=  0.2483727D 02      RSIGMA=  0.0                 H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------------
   X1=  0.52427819D 01     X2=  0.24852735D 01
     THE CONSTRAINT VALUES
      0.2425232D-05         -0.4847195D-05          0.1002975D 02          0.3742782D 01


    1ST  ORDER ESTIMATES
      F=  0.2483731D 02       P=  0.2488824D 02      G=  0.2483727D 02      RSIGMA=  0.0                 H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------------
   X1=  0.52427672D 01     X2=  0.24852760D 01
     THE CONSTRAINT VALUES
```

```
 LAGRANGE MULTIPLIERS
     F=  0.2483731D 02      P=  0.2488824D 02      G=  0.2483727D 02      RSIGMA=  0.0                  H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------
  X1=  0.524276720 01    X2=  0.248527600 01
   THE CONSTRAINT VALUES
   0.35269580 01          0.65085180 01          0.99693700-03          0.26730820-02

SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
*****************************************************

PROBLEM PARAMETERS ARE
=====================

 C5          C6

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.99476D 00  0.68074D 00

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.99828D 00  0.68363D 00

RESPECTIVE ELASTICITIES ARE :
============================

 -0.19047D 00 -0.19566D 00

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.28570D 00 -0.14761D 01

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.28565D 00 -0.14830D 01

RESPECTIVE ELASTICITIES ARE :
============================

 -0.11489D 00  0.89468D 00

*******************************************
23       POINT=  17        DOTT=  0.4338981D-06      RHO=  0.1000000D-02      MAGNITUDE=  0.1529174D 00      PHASE= 2
     F=  0.2483918D 02      P=  0.2485251D 02      G=  0.2483518D 02      RSIGMA=  0.1332714D-01   H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------
  X1=  0.524260290 01    X2=  0.248538000 01
   THE CONSTRAINT VALUES
   0.28400250-03          0.15286360-03          0.10029860 02          0.37426030 01

   2ND  ORDER ESTIMATES
     F=  0.2483718D 02      P=  0.2484184D 02      G=  0.2483716D 02      RSIGMA=  0.0                  H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
```

```
    THE CONSTRAINT VALUES
     0.1104730D-06      -0.8866205D-06       0.1002977D 02       0.3742781D 01

   1ST  ORDER ESTIMATES
      F=  0.2483718D 02    P=  0.2484230D 02    G=  0.2483718D 02    RSIGMA=  0.0              H=  0.0
    THE CURRENT VALUE CF THE PROB-M(I.E.X(I) ) VARS.ARE;
 ---------------------------------------------------------
   X1=  0.5242781OD 01    X2=  0.24852773D 01
    THE CONSTRAINT VALUES
     0.4258959D-06      -0.8593484D-06       0.1002976D 02       0.3742781D 01

  LAGRANGE MULTIPLIERS
      F=  0.2483718D 02    P=  0.2484230D 02    G=  0.2483718D 02    RSIGMA=  0.0              H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
 ---------------------------------------------------------
   X1=  0.52427810D 01    X2=  0.24852773D 01
    THE CONSTRAINT VALUES
     0.3521096D 01       0.6541779D 01       0.9970228D-04       0.2671937D-03

 SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
 *****************************************************

 PROBLEM PARAMETERS ARE
 ======================

   C5          C6

 THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
 -------------------------------------------------------------------------------------------

 -0.99185D 00  0.71022D 00

 THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
 -------------------------------------------------------------------------------------------

 -0.99153D 00  0.71350D 00

 RESPECTIVE ELASTICITIES ARE :
 =============================

  -0.18913D 00 -0.20414D 00

 THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
 -------------------------------------------------------------------------------------------

 -0.28389D 00 -0.15406D 01

 THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
 -------------------------------------------------------------------------------------------

 -0.28369D 00 -0.15478D 01

 RESPECTIVE ELASTICITIES ARE :
 =============================

  -0.11414D 00  0.93413D 00

 ****************************************
       POINT=  19      DOTT=  0.8083987D-06      RHO=  0.10000000D-03    MAGNITUDE=  0.8201477D 00     PHASE= 2
```

```
 THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
  X1=  0.524276280 01     X2=  0.248528840 01
    THE CONSTRAINT VALUES
     0.2882452D-04        0.1542979D-04        0.1002978D 02        0.3742763D 01

   2ND  ORDER ESTIMATES
      F=  0.24837190 02      P=  0.24837650 02      G=  0.24837190 02      RSIGMA=  0.0                H=  0.0
    THE CURRENT VALUE CF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
  X1=  0.524278060 01     X2=  0.248527820 01
    THE CONSTRAINT VALUES
     0.4708845D-06        0.1697422D-06        0.1002977D 02        0.3742781D 01

   1ST  ORDER ESTIMATES
      F=  0.24837190 02      P=  0.24837700 02      G=  0.24837190 02      RSIGMA=  0.0                H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
  X1=  0.524278060 01     X2=  0.248527820 01
    THE CONSTRAINT VALUES
     0.4704346D-06        0.1594513D-06        0.1002977D 02        0.3742781D 01

 LAGRANGE MULTIPLIERS
      F=  0.24837190 02      P=  0.24837700 02      G=  0.24837190 02      RSIGMA=  0.0                H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
  X1=  0.524278060 01     X2=  0.248527820 01
    THE CONSTRAINT VALUES
     0.3469268D 01        0.6480970D 01        0.9970311D-05        0.2671823D-04

SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
*******************************************************

PROBLEM PARAMETERS ARE
=====================

  C5          C6

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.811190 00  0.658200 00

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.791120 00  0.652420 00

RESPECTIVE ELASTICITIES ARE :
=============================

 -0.150900 00 -0.186660 00

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.232100 00 -0.142780 01

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------
```

```
-0.22634D 00 -0.14153D 01

RESPECTIVE ELASTICITIES ARE :
===========================

 -0.91074D-01  0.85420D 00

*********************************************
25       POINT=  21        DOTT=  0.63905740D-06       RHO=  0.10000000D-04       MAGNITUDE=  0.25509340 01       PHASE= 2
       F=  0.24837210 02       P=  0.24837430 02       G=  0.24837170 02       RSIGMA=  0.22514500D-03       H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
   X1=  0.524277910 01     X2=  0.24852791D 01
     THE CONSTRAINT VALUES
     0.28367280-05        0.15659290-05        0.10029770 02        0.37427790 01

  2ND  ORDER ESTIMATES
       F=  0.24837180 02       P=  0.24837230 02       G=  0.24837180 02       RSIGMA=  0.0                    H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
   X1=  0.524278090 01     X2=  0.248527810 01
     THE CONSTRAINT VALUES
    -0.56079490D-07       0.24148130-07        0.10029770 02        0.37427810 01

  1ST  ORDER ESTIMATES
       F=  0.24837180 02       P=  0.24837240 02       G=  0.24837180 02       RSIGMA=  0.0                    H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
   X1=  0.524278090 01     X2=  0.248527810 01
     THE CONSTRAINT VALUES
    -0.50814350-07        0.25501160-07        0.10029770 02        0.37427810 01

 LAGRANGE MULTIPLIERS
       F=  0.24837180 02       P=  0.24837240 02       G=  0.24837180 02       RSIGMA=  0.0                    H=  0.0
     THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
   X1=  0.524278090 01     X2=  0.248527810 01
     THE CONSTRAINT VALUES
     0.35251880 01        0.63859850 01        0.99703190-06        0.26718110-05

SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
*****************************************************

PROBLEM PARAMETERS ARE
=====================

  C5         C6

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
--------------------------------------------------------------------------------------------

-0.594650 00  0.529210 00

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
--------------------------------------------------------------------------------------------

-0.570590 00  0.514870 00

RESPECTIVE ELASTICITIES ARE :
-----------------------------
```

```
 -0.10883D 00 -0.14731D 00

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.17013D 00 -0.11480D 01

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.16325D 00 -0.11169D 01

RESPECTIVE ELASTICITIES ARE :
=============================

 -0.65687D-01  0.67412D 00
```

EXAMPLE-1 : SOLUTION AND SENSITIVITY ANALYSIS

SYMPUT PARAMETER CARD

N= 2 NN= 6 M= 4 MZ= 0 ITEST= 0 IPR= 0 ITES= 0 NSWIT= 0 IID= 6

TYPE NAME	TYPE #	TYPE DESCRIP.	VAR. NAME	NUMBR. ASSND.	ARG. VARIAB. NAME	NUMBR.	FUNCT. TYPE	NUMBR. ASSND.	CONSTANT C1	CONSTANT C2
V		ORIG. VARIABLE	X1	1	0.20000D 01					
V		ORIG. VARIABLE	X2	2	0.20000D 01					
P		SENSITIVITY PAR	C5					1	0.10000D 01	
P		SENSITIVITY PAR	C6					2	-0.15000D 01	
D		DATA DEFINITION	C1						0.0	
D		DATA DEFINITION	C2						0.0	
D		DATA DEFINITION	C3						0.30000000D 01	
D		DATA DEFINITION	C4						0.15000000D 01	
S	3	SEPARABL	X3	3	X1	1	SIN	20	0.50000000D 00	-.50000000D 01
S	3	SEPARABL	X3	3	X2	2	LIN	3	0.10000000D 01	0.0
									C5	
S	3	SEPARABL	X4	4	X2	2	LIN	3	0.10000000D 01	0.0
S	3	SEPARABL	X4	4	X1	1	LNG	15	0.10000000D 01	-.15000000D 01
										C6
Q	4	QUADRATC	X5	5	X1	1	LIN	3	0.10000000D 01	0.0
Q	4	QUADRATC	X5	5	X2	2	LIN	3	0.10000000D 01	0.0
S	3	SEPARABL	X6	6	X1	1	QDR	33	0.10000000D 01	0.10000000D 02
S	3	SEPARABL	X6	6	X2	2	QDR	33	0.10000000D 01	0.10000000D 01

ARRAY OF CONSTANTS CC1(.)
****

LOCATION	VALUE
1	0.10000D 01
2	-0.15000D 01
3	0.0
4	0.0
5	0.30000D 01
6	0.15000D 01
7	0.0
8	0.10000D 01
9	0.50000D 00
10	-0.50000D 01
11	0.10000D 02

BOUNDS:NCONC(.,.) : LOWER BOUND HAS MINUS SIGN

	CVF	CVF#	VALUE OF BOUND	LOCATION IN CC1(.)
1	X3	3	0.0	3
2	X4	4	0.0	-4
3	X5	5	0.300000D 01	-5
4	X1	1	0.150000D 01	-6

RELATIVE LOCATION:ITAB(.,.)

CVF#	SEPARABLE	QUADRATIC
3	1	1
4	3	1
5	5	1
6	5	2
7	7	2

```
SEPARABLE DEFINITIONS : IEPTAB(...)
  ARG   ARG#  TRAN#  TRANSF       C1    LOC.IN      C2   LOC.IN
                                          CC1                CC1
1  X1     1    20     SIN  0.5000D 00    9   -.5000D 01   10
2  X2     2     3     LIN  0.1000D 01    1    0.0          7
3  X2     2     3     LIN  0.1000D 01    8    0.0          7
4  X1     1    15     LNG  0.1000D 01    8   -.1500D 01    2
5  X1     1    33     QDR  0.1000D 01    8    0.1000D 02  11
6  X2     2    33     QDR  0.1000D 01    8    0.1000D 01   8

QUADRATIC DEFINITIONS : IUADTB(.,.) : EACH LINE REFERS TO A PAIR OF ARGUMENTS
************************ FIRST ARGUMENT ******************      ************************SECOND ARGUMENT ******************
 ARG   ARG#  TRAN#  TRANSF      C1   LOC.IN      C2   LOC.IN     ARG   ARG#  TRAN#  TRANSF      C1   LOC.IN      C2  LOC.IN
                                        CC1                CC1
1  X1     1     3    LIN  0.1000D 01    8   0.0             7 *   X2     2     3    LIN  0.1000D 01    8   0.0           7
```

```
NUNLINEAR PROGRAMMING RUUTINE-SUMT VERSION 4 08/10/71

   N=  2       M=  4       MZ=  0

     MAX. TIME= 0.50000000 03    R= 0.10000000 03    RATIO= 0.10000000 02       EPSILON= 0.1000000D-04    THETA= 0.10000000D-04

OPTIONS SELECTED
   3      2      1      1      1      1      1      2      1      1

TOLERANCES
   0.10000000D-02       0.1000000D-02

SECOND SET OF OPTIONS
   1      1
     F=  0.65000000 02      P=  0.0                G=  0.0                    RSIGMA=  0.0                H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-------------------------------------------------------------
  X1=  0.200000000D 01    X2=  0.200000000D 01
   THE CONSTRAINT VALUES
    0.22073550 01        0.96027920 00        0.10000000 01        0.50000000 00

*****THE FEASIBLE STARTING POINT TO BE USED IS ...
     F=  0.65000000 02    P=  0.0                 G=  0.0                    RSIGMA=  0.0                H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-------------------------------------------------------------
  X1=  0.200000000D 01    X2=  0.200000000D 01
   THE CONSTRAINT VALUES
    0.22073550 01        0.96027920 00        0.10000000 01        0.50000000 00
    1                  0.200000000D 01
    2                  0.200000000D 01
A   3                 -0.220735490 01
    4                  0.960279230 00
    5                  0.400000000D 01
    6                  0.650000000 02
  1
    -0.1350760 01         0.1000000 01
  2
B   -0.7500000 00         0.1000000 01
  3
     0.2000000 01         0.2000000 01
  4
    -0.1600000 02         0.2000000 01
  1
     0.1051840 01         0.0
  2
C    0.3750000 00         0.0
  3
     0.0                  0.0
  4
     0.2000000 01         0.2000000 01
  1     1
  2     1
D 3     1
  1     2          0.1000000 01
  4     2
```

```
                         EXAMPLE-1 : SOLUTION AND SENSITIVITY ANALYSTS
SYMPUT PARAMETER CARD
N=  2   NN=  6   M=  4   MZ=  0   ITEST=  1   IPR=  0   ITES=  2   NSWIT=  0   IIC=  6

 TYPE  TYPE  TYPE      VAR.  NUMBR.    ARG.     NUMBR.      FUNCT. NUMBR.  CONSTANT      CONSTANT
 NAME   #    DESCRIP.  NAME  ASSND.    VARIAB.               TYPE  ASSND.     C1            C2
                                       NAME

 V   ORIG. VARIABLE     X1     1   0.20000D 01
 V   ORIG. VARIABLE     X2     2   0.20000D 01

 P  SENSITIVITY PAR     C5                                            1    0.10000D 01

 P  SENSITIVITY PAR     C6                                            2   -0.15000D 01
 D  DATA DEFINITION     C1                                                0.0
 D  DATA DEFINITION     C2                                                0.0
 D  DATA DEFINITION     C3                                                0.300000000 01
 D  DATA DEFINITION     C4                                                0.150000000 01
 S      3    SEPARABL   X3     3         X1        1         SIN       20 0.500000000 00  -.500000000 01
 S      3    SEPARABL   X3     3         X2        2         LIN        3 0.100000000 01  0.0
                                                                           C5
 S      3    SEPARABL   X4     4         X2        2         LIN        3 0.100000000 01  0.0
 S      3    SEPARABL   X4     4         X1        1         LNG       15 0.100000000 01  -.150000000 01
                                                                                          C6
 Q      4    QUADRATC   X5     5         X1        1         LIN        3 0.100000000 01  0.0
 Q      4    QUADRATC   X5     5         X2        2         LIN        3 0.100000000 01  0.0
 S      3    SEPARABL   X6     6         X1        1         QDR       33 0.100000000 01  0.100000000 02
 S      3    SEPARABL   X6     6         X2        2         QDR       33 0.100000000 01  0.100000000 01

   ARRAY OF CONSTANTS CC1(.)
   ********************

     LOCATION    VALUE

          1      0.10000D 01
          2     -0.15000D 01
          3      0.0
          4      0.0
          5      0.30000D 01
          6      0.15000D 01
          7      0.0
          8      0.10000D 01
          9      0.50000D 00
         10     -0.50000D 01
         11      0.10000D 02

   BOUNDS:NCONC(...)  : LOWER BOUND HAS MINUS SIGN

   CVF   CVF#  VALUE OF     LOCATION
                 BOUND      IN CC1(.)
 1   X3    3  0.0              3
 2   X4    4  0.0             -4
 3   X5    5  0.30000D 01     -5
 4   X1    1  0.15000D 01     -6

   RELATIVE LOCATION:ITAB(...)
 CVF# SEPARABLE QUADRATIC
  3       1         1
  4       3         1
  5       5         1
  6       5         2
  7       7         2
```

```
SEPARABLE DEFINITIONS : IEPTAB(...)
 ARG   ARG#  TRAN#  TRANSF      C1    LOC.IN      C2  LOC.IN
                                       CC1             CC1
 1  X1    1    20    SIN  0.5000D 00    9  -.5000D 01  10
 2  X2    2     3    LIN  0.1000D 01    1  0.0          7
 3  X2    2     3    LIN  0.1000D 01    8  0.0          7
 4  X1    1    15    LNG  0.1000D 01    8  -.1500D 01   2
 5  X1    1    33    QDR  0.1000D 01    8  0.1000D 02  11
 6  X2    2    33    QDR  0.1000D 01    8  0.1000D 01   8

QUADRATIC DEFINITIONS : IUADTB(...) : EACH LINE REFERS TO A PAIR OF ARGUMENTS
************************** FIRST ARGUMENT *******************     **************************SECOND ARGUMENT *******************
 ARG   ARG#  TRAN#  TRANSF      C1   LOC.IN      C2   LOC.IN     ARG   ARG#  TRAN#  TRANSF      C1   LOC.IN      C2  LOC.IN
                                      CC1               CC1
 1  X1    1     3    LIN  0.1000D 01    8  0.0             7  *   X2     2     3    LIN  0.1000D 01    8  0.0          7
```

```
NONLINEAR PROGRAMMING ROUTINE-SUMT VERSION 4 08/10/71

   N=  2      M=  4      MZ=  0

     MAX. TIME= 0.5000000D 03    R= 0.1000000D 03    RATIO= 0.1000000D 02      EPSILON= 0.1000000D-04    THETA= 0.1000000D-04

OPTIONS SELECTED
   3      2      1      1      1      1      1      2      1      1

TOLERANCES
   0.1000000D-02       0.1000000D-02

SECOND SET OF OPTIONS
   1      1
       F=  0.6500000D 02       P=  0.0                    G=  0.0                   RSIGMA=  0.0                H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
  X1=  0.200000000 01    X2=  0.200000000 01
   THE CONSTRAINT VALUES
    0.2207355D 01        0.9602792D 00        0.1000000D 01        0.5000000D 00

*****THE FEASIBLE STARTING POINT TO BE USED IS ...
       F=  0.6500000D 02       P=  0.0                    G=  0.0                   RSIGMA=  0.0                H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
  X1=  0.200000000D 01    X2=  0.200000000D 01
   THE CONSTRAINT VALUES
    0.2207355D 01        0.9602792D 00        0.1000000D 01        0.5000000D 00

********************************************
        POINT=    3         DOTT=  0.9541129D-05      RHO=  0.1000000D 03        MAGNITUDE=  0.3156616D-01       PHASE= 2
       F=  0.4609753D 02       P= -0.3344122D 03      G= -0.3539025D 03      RSIGMA= -0.3805097D 03      H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
  X1=  0.375055370 01    X2=  0.365366660 01
   THE CONSTRAINT VALUES
    0.1116348D 01        0.1670811D 01        0.1070327D 02        0.2250554D 01

  LAGRANGE MULTIPLIERS
       F=  0.4609753D 02       P= -0.3344122D 03      G= -0.3539025D 03      RSIGMA= -0.3805097D 03      H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
  X1=  0.375055370 01    X2=  0.365366660 01
   THE CONSTRAINT VALUES
    0.8957784D 02        0.5985116D 02        0.9342937D 01        0.4443351D 02

SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
*****************************************************

PROBLEM PARAMETERS ARE
======================

  C5            C6

 THE VALUES OF SIG(J) IN DLPEN ARE:
E 0.100000000 01  0.0

 THE VALUES OF ALF RETURNED BY INVERS ARE :
F 0.6000808lD-02 -0.22297426D-02

 IN EXOFLD : THE VALS. OF CVES XX(I) ARE :
```

```
G  6  0.460975260 02

   THE VALS. OF DM(J)      : I.E. PARTIALS OF NEW CVFS W.R.T. ORIGINAL PROB. VARS. (N BY 1) PREMULT. BY ALF ;

H 0.226782740-02 -0.462971150-02  0.135621830-01 -0.108014550 00

   THE VALS. OF DIAGA (I,J) I.E. SECOND PARTIALS OF NEW CVFS W.R.T. Y    & X PREMULT. BY ALF;

   3 -0.222974260-02  0.0
I  4  0.0             0.159997930-02
   5  0.0             0.0
   6  0.0             0.0

  THE VALS OF DELEX(I,J) ;PARTIALS OF NEW CVF WRT SENS VAR. Y;

   3  0.365366660 01  0.0
J  4  0.0             0.132190350 01
   5  0.0             0.0
   6  0.0             0.0

K   THE VALUES OF DELA(J) ARE:
  0.465139590 00 -0.314990030 00

  THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF   X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
  ------------------------------------------------------------------------------------------

  -0.465140 00  0.314990 00

  RESPECTIVE ELASTICITIES ARE :
  =============================

   -0.124020 00 -0.125980 00

   THE VALUES OF SIG(J) IN DLPEN ARE:
   0.0          0.100000000 01

   THE VALUES OF ALF RETURNED BY INVERS ARE :
  -0.222974260-02  0.850058150-02

   IN EXDELP : THE VALS. OF CVFS XX(J) ARE ;
    1  0.375055370 01      2  0.365366660 01       3  -.111634750 01       4  0.167081140 01       5  0.137032730 02
    6  0.460975260 02

   THE VALS. OF DM(J)      : I.E. PARTIALS OF NEW CVFS W.R.T. ORIGINAL PROB. VARS. (N BY 1) PREMULT. BY ALF ;

   0.682940270-02  0.939234690-02  0.237351510-01  0.401353670-01

   THE VALS. OF DIAGA (I,J) I.E. SECOND PARTIALS OF NEW CVFS W.R.T. Y    & X PREMULT. BY ALF;

   3  0.850058150-02  0.0
   4  0.0            -0.594510250-03
   5  0.0             0.0
   6  0.0             0.0

  THE VALS OF DELEX(I,J) ;PARTIALS OF NEW CVF WRT SENS VAR. Y;

   3  0.365366660 01  0.0
   4  0.0             0.132190350 01
   5  0.0             0.0
   6  0.0             0.0

    THE VALUES OF DELA(J) ARE:
   0.276368840 01  0.480335270 00
```

EXAMPLE-2 : SOLUTION AND SENSITIVITY ANALYSIS OF HOPKINS PROBLEM

SYMPUT PARAMETER CARD
N= 8 NN= 14 M= 12 MZ= 0 ITEST= 1 IPR= 0 ITES= 0 NSWIT= 0 IIC= 9

TYPE NAME	TYPE #	TYPE DESCRIP.	VAR. NAME	NUMBR. ASSND.	ARG. VARIAB. NAME	NUMBR.	FUNCT. TYPE	NUMBR. ASSND.	CONSTANT C1	CONSTANT C2
V	ORIG.	VARIABLE	X1	1	0.18000D 03					
V	ORIG.	VARIABLE	X2	2	0.74000D 03					
V	ORIG.	VARIABLE	X3	3	0.26800D 04					
V	ORIG.	VARIABLE	X4	4	0.14650D 05					
V	ORIG.	VARIABLE	X5	5	0.40000D 00					
V	ORIG.	VARIABLE	X6	6	0.23000D 01					
V	ORIG.	VARIABLE	X7	7	0.60000D 00					
V	ORIG.	VARIABLE	X8	8	0.20000D 00					
P	SENSITIVITY	PAR	A1					1	0.24000D 02	
P	SENSITIVITY	PAR	A2					2	0.12000D 02	
P	SENSITIVITY	PAR	A3					3	0.60000D 00	
P	SENSITIVITY	PAR	A4					4	0.40000D 00	
P	SENSITIVITY	PAR	A5					5	0.10000D 00	
P	SENSITIVITY	PAR	A6					6	0.15000D 00	
P	SENSITIVITY	PAR	A7					7	-0.15000D 00	
P	SENSITIVITY	PAR	A8					8	0.11000D 02	
P	SENSITIVITY	PAR	A9					9	-0.24000D 02	
Q	4	QUADRATC	X9	9	X4	4	IDN	2	0.0	0.0
Q	4	QUADRATC	X9	9	X6	6	IDN	2	0.0	0.0
Q	4	QUADRATC	X10	10	X3	3	IDN	2	0.0	0.0
Q	4	QUADRATC	X10	10	X6	6	IDN	2	0.0	0.0
S	3	SEPARABL	X11	11	X1	1	LIN	3	0.24000000D 02	0.0
									A1	
S	3	SEPARABL	X11	11	X2	2	LIN	3	0.12000000D 02	0.0
									A2	
S	3	SEPARABL	X11	11	X3	3	LIN	3	0.60000000D 00	0.0
									A3	
S	3	SEPARABL	X11	11	X4	4	LIN	3	0.40000000D 00	0.0
									A4	
S	3	SEPARABL	X11	11	X10	10	LIN	3	0.10000000D 00	0.0
									A5	
S	3	SEPARABL	X11	11	X9	9	LIN	3	0.15000000D 00	0.0
									A6	
Q	4	QUADRATC	X11	11	X9	9	LIN	3	-.15000000D 00	0.0
									A7	
Q	4	QUADRATC	X11	11	X5	5	IDN	2	0.0	0.0
Q	4	QUADRATC	X11	11	X1	1	LIN	3	0.11000000D 02	0.0
									A8	
Q	4	QUADRATC	X11	11	X7	7	IDN	2	0.0	0.0
Q	4	QUADRATC	X11	11	X1	1	LIN	3	-.24000000D 02	0.0
									A9	
Q	4	QUADRATC	X11	11	X8	8	IDN	2	0.0	0.0
S	3	SEPARABL	X12	12	X1	1	LIN	3	0.12250000D 02	0.0
S	3	SEPARABL	X12	12	X10	10	IDN	2	0.0	0.0
S	3	SEPARABL	X12	12	X9	9	LIN	3	0.70000000D 00	0.0

Q	4	QUADRATC	X12	12	X[illegible]	[illegible]				
Q	4	QUADRATC	X12	12	X5	5	IDN	2	0.0	0.0
S	3	SEPARABL	X13	13	X6	6	[illegible]	33	0.100000000D 01	0.500000000 01
S	3	SEPARABL	X14	14	X1	1	PWR	11	0.500000000D 00	-.100000000 03
S	3	SEPARABL	X14	14	X2	2	LIN	3	-.200000000D 01	0.0
S	3	SEPARABL	X14	14	X3	3	PWR	11	0.500000000D 00	-.250000000 02
S	3	SEPARABL	X14	14	X4	4	PWR	11	0.500000000D 00	-.500000000 02
S	3	SEPARABL	X14	14	X5	5	LIN	3	-.100000000D 02	0.0
S	3	SEPARABL	X14	14	X13	13	LIN	3	0.100000000D 03	0.0
Q	4	QUADRATC	X14	14	X1	1	PWR	11	0.500000000D 00	-.300000000 02
Q	4	QUADRATC	X14	14	X7	7	PWR	10	0.500000000D 00	0.0
S	3	SEPARABL	X14	14	X8	8	LIN	3	0.100000000D 03	0.0
Q	4	QUADRATC	X14	14	X1	1	LIN	3	-.100000000D 02	0.0
Q	4	QUADRATC	X14	14	X3	3	PWR	10	-.100000000D 01	0.0
Q	4	QUADRATC	X14	14	X1	1	LIN	3	-.200000000D 02	0.0
Q	4	QUADRATC	X14	14	X4	4	PWR	10	-.100000000D 01	0.0

ARRAY OF CONSTANTS CC1(.)
****

LOCATION	VALUE
1	0.240000D 02
2	0.120000D 02
3	0.500000D 00
4	0.400000D 00
5	0.100000D 00
6	0.150000D 00
7	-0.150000D 00
8	0.110000D 02
9	-0.240000D 02
10	0.0
11	0.100000D 01
12	0.122500D 02
13	0.700000D 00
14	-0.200000D 00
15	0.300000D 01
16	0.500000D 00
17	-0.100000D 03
18	-0.200000D 01
19	0.500000D 00
20	-0.250000D 02
21	0.500000D 00
22	-0.500000D 02
23	-0.100000D 02
24	0.100000D 03
25	0.500000D 00
26	-0.300000D 02
27	0.500000D 00
28	0.100000D 03
29	-0.100000D 02
30	-0.100000D 01
31	-0.200000D 02
32	-0.100000D 01
33	0.250000D 05
34	0.340000D 05
35	0.500000D 00
36	0.300000D 00

BOUNDS:NC,INC(...) : LOWER BOUND HAS MINUS SIGN

CVF		CVF#	VALUE OF BOUND	LOCATION IN CC1(.)
1	X11	11	0.250000D 05	33

5	X1	1	0.0	-10
6	X2	2	0.0	-10
7	X3	3	0.0	-10
8	X4	4	0.0	-10
9	X5	5	0.0	-10
10	X6	6	0.0	-10
11	X7	7	0.0	-10
12	X8	8	0.0	-10

RELATIVE LOCATION:ITAB(...)

CVF#	SEPARABLE	QUADRATIC
9	1	1
10	1	2
11	1	3
12	7	6
13	10	7
14	11	7
15	18	10

SEPARABLE DEFINITIONS : IEPTAB(...)

	ARG	ARG#	TRAN#	TRANSF	C1	LOC.IN CC1	C2	LOC.IN CC1
1	X1	1	3	LIN	0.24000D 02	1	0.0	10
2	X2	2	3	LIN	0.12000D 02	2	0.0	10
3	X3	3	3	LIN	0.80000D 00	3	0.0	10
4	X4	4	3	LIN	0.40000D 00	4	0.0	10
5	X10	10	3	LIN	0.10000D 00	5	0.0	10
6	X9	9	3	LIN	0.15000D 00	6	0.0	10
7	X1	1	3	LIN	0.12250D 02	12	0.0	10
8	X10	10	2	IDN	0.0	10	0.0	10
9	X9	9	3	LIN	0.70000D 00	13	0.0	10
10	X6	6	33	QDR	0.10000D 01	11	0.3000D 01	15
11	X1	1	11	POW	0.50000D 00	16	-.1000D 03	17
12	X2	2	3	LIN	-.20000D 01	18	0.0	10
13	X3	3	11	POW	0.50000D 00	19	-.2500D 02	20
14	X4	4	11	POW	0.50000D 00	21	-.5000D 02	22
15	X5	5	3	LIN	-.10000D 02	23	0.0	10
16	X13	13	3	LIN	0.10000D 03	24	0.0	10
17	X8	8	3	LIN	0.10000D 03	28	0.0	10

QUADRATIC DEFINITIONS : IQADTB(...) : EACH LINE REFERS TO A PAIR OF ARGUMENTS

************************** FIRST ARGUMENT ********************* ****************************SECOND ARGUMENT *********************

	ARG	ARG#	TRAN#	TRANSF	C1	LOC.IN CC1	C2	LOC.IN CC1		ARG	ARG#	TRAN#	TRANSF	C1	LOC.IN	C2	LOC.IN
1	X4	4	2	IDN	0.0	10	0.0	10	*	X6	6	2	IDN	0.0	10	0.0	10
2	X3	3	2	IDN	0.0	10	0.0	10	*	X6	6	2	IDN	0.0	10	0.0	10
3	X9	9	3	LIN	-.15000D 00	7	0.0	10	*	X5	5	2	IDN	0.0	10	0.0	10
4	X1	1	3	LIN	0.11000D 02	8	0.0	10	*	X7	7	2	IDN	0.0	10	0.0	10
5	X1	1	3	LIN	-.24000D 02	9	0.0	10	*	X6	6	2	IDN	0.0	10	0.0	10
6	X9	9	3	LIN	-.20000D 03	14	0.0	10	*	X5	5	2	IDN	0.0	10	0.0	10
7	X1	1	11	POW	0.50000D 00	25	-.3000D 02	26	*	X7	7	11	POW	0.50000D 00	27	0.0	10
8	X1	1	3	LIN	-.10000D 02	29	0.0	10	*	X3	3	10	[illegible]	-.10000D 01	30	0.0	10
9	X1	1	3	LIN	-.20000D 02	31	0.0	10	*	X4	4	10	[illegible]	-.10000D 01	32	0.0	10

```
NONLINEAR PROGRAMMING ROUTINE-SUMT VERSION 4 10/10/71

    N=  8        M= 12        MZ=  0

      MAX. TIME= 0.30000000 03    K= 0.10000000 03    RATIO= 0.10000000 02    EPSILON= 0.10000000-04    THETA= 0.10000000-04

OPTIONS SELECTED
    3      2      2      1      1      1      1      2      1      1

TOLERANCES
    0.10000000-02       0.10000000-02

SECOND SET OF OPTIONS
    1      1
      F= -0.10415400 05      P=  0.0            G=  0.0                RSIGMA=  0.0                    H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------------------
  X1=  0.18300000D 03    X2=  0.74000000D 03    X3=  0.26800000D 04    X4=  0.14650000D 05    X5=  0.40000000D 01
  X6=  0.23000000D 01    X7=  0.60000000D 00    X8=  0.20000000D 00
    THE CONSTRAINT VALUES
     0.35905000D 03      -0.47401000D 04      0.10000000D 00      0.10000000D 00      0.18300000D 03      0.74000000D 03
     0.26800000D 04       0.14650000D 05      0.40000000D 01      0.23000000D 01      0.60000000D 00      0.20000000D 00

     ORTHOGONAL MOVE

     DEL P VECTOR
  0.56199268D 01  -0.32073178D 01   0.21061479D 01   0.12637668D 01  -0.30813270D 04   0.11364618D 05  -0.33473636D 03
  0.70317505D 03

     SECOND ORDER MOVE VECTOR
 -0.12359123D 04  -0.17518576D 05  -0.84804364D 05   0.49877509D 06  -0.26767502D 01  -0.93633714D 00  -0.10000000D 01
  0.0

**************************************************
        POINT=    1         DUTT=  0.50653090 06       RHO=  0.10000000 01       MAGNITUDE=  0.12714320 05       PHASE= 1
      F=  0.0                   P= -0.35086010 04      G= -0.12000000 04      RSIGMA= -0.35086010 04       H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------------------
  X1=  0.15369203D 03    X2=  0.36709553D 03    X3=  0.37483454D 03    X4=  0.25267343D 05    X5=  0.34259635D 00
  X6=  0.22789918D 01    X7=  0.57871377D 00    X8=  0.20000000D 00
    THE CONSTRAINT VALUES
     0.15616390D 03       0.62392650D 04      0.15740370D 00      0.10000000D 00      0.15369210D 03      0.36709550D 03
     0.87483460D 03       0.25267340D 05      0.34259630D 00      0.22789920D 01      0.57871360D 00      0.20000000D 00

*****THE FEASIBLE STARTING POINT TO BE USED IS ...
      F= -0.10877410 05      P= -0.35086010 04      G=  0.0                RSIGMA= -0.35086010 04      H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------------------
  X1=  0.15369203D 03    X2=  0.36709553D 03    X3=  0.87483454D 03    X4=  0.25267343D 05    X5=  0.34259635D 00
  X6=  0.22789918D 01    X7=  0.57871377D 00    X8=  0.20000000D 00
    THE CONSTRAINT VALUES
     0.15616390D 03       0.62392660D 04      0.15740370D 00      0.10000000D 00      0.15369210D 03      0.36709550D 03
     0.87483460D 03       0.25267040D 05      0.34259630D 00      0.22789920D 01      0.57871380D 00      0.20000000D 00

     ORTHOGONAL MOVE

     DEL P VECTOR
 -0.10558367D 02  -0.34116270D 01   0.41295073D-01  -0.21575450D 02   0.59[illegible]D 04  -0.119[illegible]D 04  -0.[illegible]D 03
  0.17620126D 04

     SECOND ORDER MOVE VECTOR
 -0.33348553D 03   0.10[illegible]D 02  -0.33713[illegible]D 03   0.13[illegible]D 00   0.[illegible]D 00   0.11[illegible]D 01   0.1[illegible]D-01
  0.0
```

T-402

```
*********************************************
      POINT=   1        DOTT=  0.17227360 04      RHO=  0.10000000 03      MAGNITUDE=  0.53408270 04      PHASE= 2
    F= -0.11035330 05     P= -0.14551990 05     G= -0.12235080 05     RSIGMA= -0.35130830 04     H=  0.0
  THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------
  X1=  0.11400940 03    X2=  0.38934416D 01    X3=  0.81391934D 03    X4=  0.27271750D 05    X5=  0.45907578D 00
  X6=  0.24143196D 01    X7=  0.64775743D 00    X8=  0.20000000D 00
   THE CONSTRAINT VALUES
    0.32686490 03        0.93989830 04        0.40924220-01        0.10000000 01        0.11400950 03        0.33764420 03
    0.81391930 03        0.27271760 05        0.45907530 00        0.24143200 01        0.64775740 00        0.20000000 00

    DEL P VECTOR
 -0.13453008D 01  -0.14145970D 01   0.32880131D 00  -0.11634176D-01   0.56573935D 03  -0.35813177D 03  -0.48634621D 02
  0.23711263D 03

    SECOND ORDER MOVE VECTOR
  0.15391214D 02  -0.10889101D 04   0.11036594D 04   0.32236599D 05  -0.87521533D-02  -0.32863051D 01  -0.42517603D-01
  0.15288674D-01

*********************************************
      POINT=   2        DOTT=  0.26842720 04      RHO=  0.10000000 03      MAGNITUDE=  0.79376070 03      PHASE= 2
    F= -0.11603630 05     P= -0.15017960 05     G= -0.12803680 05     RSIGMA= -0.34142810 04     H=  0.0
  THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------
  X1=  0.11788964D 03    X2=  0.11512735D 03    X3=  0.10891493D 04    X4=  0.35398713D 05    X5=  0.45686934D 00
  X6=  0.15858339D 01    X7=  0.68698861D 00    X8=  0.20385432D 00
   THE CONSTRAINT VALUES
    0.91590970 03        0.33374870 04        0.43130660-01        0.96145630-01        0.11788960 03        0.11512700 03
    0.10891500 04        0.35398710 05        0.45686530 00        0.15858340 01        0.68698860 00        0.20385430 00

    DEL P VECTOR
  0.40638822D 01   0.15584327D 01   0.43427563D 00   0.10684303D 03  -0.15066965D 04   0.69729750D 03   0.20047460D 03
 -0.34063025D 03

    SECOND ORDER MOVE VECTOR
 -0.12024320D 02  -0.49141479D 02   0.11223058D 04   0.36184747D 04   0.30507843D-04  -0.15912243D 00  -0.10100062D-01
  0.62556310D-02

*********************************************
      POINT=   3        DOTT=  0.63338190 03      RHO=  0.10000000 03      MAGNITUDE=  0.17066350 04      PHASE= 2
    F= -0.11943590 05     P= -0.15238880 05     G= -0.13143590 05     RSIGMA= -0.32952940 04     H=  0.0
  THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------
  X1=  0.11145180D 03    X2=  0.88816627D 02    X3=  0.16900340D 04    X4=  0.37336050D 05    X5=  0.45688556D 00
  X6=  0.15006395D 01    X7=  0.68158102D 00    X8=  0.23720359D 00
   THE CONSTRAINT VALUES
    0.21150460 03        0.40013110 04        0.43114440-01        0.92796410-01        0.11145180 03        0.88816630 02
    0.16900340 04        0.37336050 05        0.45688560 00        0.15006430 01        0.68158130 00        0.23720360 00

    ORTHOGONAL MOVE

    DEL P VECTOR
 -0.54218274D 01  -0.25477192D 01   0.45713605D-01  -0.92037677D-01   0.16027431D 04  -0.54156124D 03  -0.24111225D 03
  0.56966285D 03

    SECOND ORDER MOVE VECTOR
 -0.12057580D 03   0.52606015D 01   0.83064542D 02   0.42743443D 04   0.11596737D-02  -0.11799232D 00   0.10000000D 01
  0.0

*********************************************
      POINT=   4        DOTT=  0.75353590 02      RHO=  0.10000000 03      MAGNITUDE=  0.19314490 04      PHASE= 2
    F= -0.11924240 05     P= -0.15255850 05     G= -0.13124240 05     RSIGMA= -0.33316070 04     H=  0.0
  THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------
  X1=  0.79933033D 02    X2=  0.93171845D 02    X3=  0.17119085D 04    X4=  0.39463417D 05    X5=  0.45718434D 00
```

```
*********************************************
      POINT=  23        DGTI=  0.4557075D-03       RHO=  0.1000000D-01       MAGNITUDE=  0.2[illegible]D 03       PHASE= 2
    F= -0.1257379D 05       P= -0.1257357D 05      G= -0.1257371D 05      RSIGMA=  0.1117780D 00       H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-------------------------------------------------------------
  X1=  0.1192159[illegible]D 03     X2=  0.[illegible]D-02     X3=  0.475722[illegible]D 04     X4=  0.3990[illegible]D 05     X5=  0.4999942[illegible]D 00
  X6=  0.1133[illegible]D 01     X7=  0.1870[illegible]D 00     X8=  0.29[illegible]D 00
   THE CONSTRAINT VALUES
    0.3432454[illegible]D-01        0.4[illegible]D 00        0.57299[illegible]D-05        0.14[illegible]D-04        0.11[illegible]D 03        0.[illegible]D-02
    0.4759222[illegible]D 04        0.3999372D 05        0.4999943[illegible]D 00        0.1133622[illegible]D 01        0.1870[illegible]D 00        0.2[illegible]D 00
APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

  2ND  ORDER ESTIMATES
    F= -0.1257384D 05       P= -0.1257372D 05      G= -0.1257[illegible]D 05      RSIGMA=  0.0                    H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-------------------------------------------------------------
  X1=  0.1192295[illegible]D 03     X2=  0.8929172[illegible]D-04     X3=  0.475919[illegible]D 04     X4=  0.3990[illegible]D 05     X5=  0.4999991[illegible]D 00
  X6=  0.1133597[illegible]D 01     X7=  0.1870180[illegible]D 00     X8=  0.2999997[illegible]D 00
   THE CONSTRAINT VALUES
    0.4333422D-03        0.5324138D-02        0.9368383D-07        0.21384[illegible]D-06        0.11922[illegible]D 03        0.8929173D-04
    0.4759190D 04        0.3999089[illegible]D 05        0.4999999D 00        0.1133597[illegible]D 01        0.1870181D 00        0.2999997[illegible]D 00

  1ST  ORDER ESTIMATES
    F= -0.1257384D 05       P= -0.1257371D 05      G= -0.1257384D 05      RSIGMA=  0.0                    H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-------------------------------------------------------------
  X1=  0.1192288D 03     X2=  0.8684010D-04     X3=  0.4759188[illegible]D 04     X4=  0.3999016D 05     X5=  0.4999999[illegible]D 00
  X6=  0.1133597[illegible]D 01     X7=  0.1870181D 00     X8=  0.2999997[illegible]D 00
   THE CONSTRAINT VALUES
    0.4153131D-03        0.4797492D-02        0.9368180D-07        0.2199293D-06        0.1192283D 03        0.8684010D-04
    0.4759188D 04        0.3999902D 05        0.4999999[illegible]D 00        0.1133597[illegible]D 01        0.1870182D 00        0.2999999[illegible]D 00

 LAGRANGE MULTIPLIERS
    F= -0.1257384D 05       P= -0.1257371D 05      G= -0.1257334D 05      RSIGMA=  0.0                    H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-------------------------------------------------------------
  X1=  0.1192288D 03     X2=  0.8684010D-04     X3=  0.4759188[illegible]D 04     X4=  0.3999016D 05     X5=  0.4999999[illegible]D 00
  X6=  0.1133597[illegible]D 01     X7=  0.1870181D 00     X8=  0.2999997[illegible]D 00
   THE CONSTRAINT VALUES
    0.2913367[illegible]D 00        0.2176033D-01        0.1745216D 04        0.7153357[illegible]D 03        0.3381[illegible]D-04        0.1452[illegible]D 01
    0.2101184D-05        0.2505713D-06        0.2000023D-01        0.8821274D-02        0.3345288D-01        0.3333489D-01

SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
*********************************************************

PROBLEM PARAMETERS ARE
=====================

 A1             A2              A3              A4              A5              A6              A7              A8              A9

      DEL P VECTOR
 0.1192369[illegible]D 03   0.694731[illegible]D-02   0.4759034[illegible]D 04   0.3[illegible]D 05   0.4999[illegible]D 00   0.1133[illegible]D 01   0.1[illegible]D 00
 0.2999855[illegible]D 00

      SECOND ORDER MOVE VECTOR
 0.4630155[illegible]D 02  -0.269[illegible]D-03  -0.17[illegible]D 03  -0.1517[illegible]D 04   0.[illegible]D-03   0.[illegible]D-01  -0.[illegible]D-01
 0.4[illegible]343970D-05

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------
```

```
[illegible] IS  0.135153D 06  0.13237D 05  0.15294D 06  0.164?D 05  0.7520D 02  0.11??D 03

THE ESTIMATES OF FIRST ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

[illegible]  0.17??D 05  0.19606D 06  0.20252D 05  0.169??D 06  0.64?20D 05  0.33?25D 02  0.13334D 03

RESPECTIVE ELASTICITIES ARE :
=============================

[illegible]  0.9935?D 02  0.53349D 05  0.16933D 02  0.213?3D 03 -0.13085D 03  0.77067D 01 -0.26173D 02

      DEL P VECTOR
[illegible]  0.6947316?D-02  0.47590473D 04  0.39908549D 05  0.49999416D 00  0.11336245D 01  0.18692913D 00

      SECOND ORDER MOVE VECTOR
[illegible]  0.48914610D-02  -0.11323661D-01  -0.10247237D 00  0.57946203D-11  0.23991252D-05  -0.69637717D-07

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

[illegible] -0.12766D-02 -0.53772D-01 -0.44872D 00 -0.65103D-01 -0.54344D 00 -0.27172D 00 -0.25035D-03 -0.40532D-03

THE ESTIMATES OF FIRST ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

[illegible]  0.18151D-03 -0.39142D-01 -0.32834D 00 -0.43928D-01 -0.36850D 00 -0.18426D 00 -0.18163D-03 -0.29345D-03

RESPECTIVE ELASTICITIES ARE :
=============================

[illegible]  0.22922D 00 -0.34112D 01 -0.19077D 02 -0.63305D 00 -0.80287D 01  0.40145D 01 -0.29320D 00  0.10230D 01

      DEL P VECTOR
[illegible]  0.69473165D-02  0.47590473D 04  0.39908549D 05  0.49999416D 00  0.11336245D 01  0.18692913D 00
[illegible]

      SECOND ORDER MOVE VECTOR
[illegible]  -0.11323661D-01  0.45639528D 05  -0.57590843D 05  -0.43554402D-05  -0.36275522D 00  -0.38112488D-01
[illegible]

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X3  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

[illegible] -0.70895D-01 -0.64535D 05 -0.41330D 06 -0.72712D 05 -0.46431D 05 -0.23215D 06 -0.22945D 03 -0.37034D 03

THE ESTIMATES OF FIRST ORDER SENSITIVITY OF  X3  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

[illegible] -0.77459D-01 -0.69879D 05 -0.45797D 06 -0.78736D 05 -0.51495D 05 -0.25747D 06 -0.25436D 03 -0.41310D 03

RESPECTIVE ELASTICITIES ARE :
=============================

[illegible] -0.19531D-03 -0.88122D 01 -0.33491D 02 -0.16544D 01 -0.16253D 02  0.81150D 01 -0.59?7?D 00  0.20681D 01

      DEL P VECTOR
[illegible]  0.69473165D-02  0.47590473D 04  0.39908549D 05  0.49999416D 00  0.11336245D 01  0.18692913D 00
[illegible]

      SECOND ORDER MOVE VECTOR
```

```
-0.66225453D-04

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X4  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.10308D 05 -0.59095D 00 -0.41237D 06 -0.36422D 07 -0.46632D 06 -0.41217D 07 -0.20613D 07 -0.19129D 04 -0.33922D 04
THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X4  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.11404D 05 -0.63614D 00 -0.45610D 06 -0.40087D 07 -0.51619D 06 -0.45372D 07 -0.22696D 07 -0.21174D 04 -0.37210D 04
RESPECTIVE ELASTICITIES ARE :
============================

 -0.68579D 01 -0.19128D-03 -0.68571D 01 -0.40179D 02 -0.12934D 01 -0.17053D 02  0.35265D 01 -0.58362D 00  0.20573D 01

      DEL P VECTOR
  0.11923698D 03    0.69473165D-02    0.47599473D 04    0.39903549D 05    0.49999416D 03    0.11336245D 01    0.13692913D 00
  0.29998561D 00

      SECOND ORDER MOVE VECTOR
  0.69208746D-08    0.57986200D-11   -0.43554402D-05    0.44672237D-04    0.27721210D-08   -0.15539430D-10    0.13401591D-09
 -0.41922338D-14

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X5  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.30116D-06 -0.18965D-10 -0.10673D-04 -0.11316D-03 -0.12078D-04 -0.12810D-03 -0.10659D-05 -0.50997D-07 -0.90345D-07
THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X5  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.82807D-06 -0.28053D-09 -0.33349D-04 -0.27449D-03 -0.37813D-04 -0.31123D-03 -0.14924D-03 -0.15425D-06 -0.24835D-06
RESPECTIVE ELASTICITIES ARE :
============================

 -0.39748D-04 -0.67328D-08 -0.40019D-04 -0.21960D-03 -0.75623D-05 -0.93371D-04  0.44774D-04 -0.33936D-05  0.11721D-04

      DEL P VECTOR
  0.11923698D 03    0.69473165D-02    0.47599473D 04    0.39908549D 05    0.49999416D 00    0.11336245D 01    0.13692913D 00
  0.29998561D 00

      SECOND ORDER MOVE VECTOR
  0.23138552D-01    0.29901252D-05   -0.36275622D 00   -0.61701676D 00   -0.15539430D-10    0.17100423D-04    0.42140128D-04
 -0.19369425D-09

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X6  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

 0.12699D 00  0.68517D-05  0.54443D 01  0.44947D 02  0.61454D 01  0.50731D 02  0.25365D 02  0.22294D-01  0.38095D-01
THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X6  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

 0.13921D 00  0.71597D-05  0.59316D 01  0.49034D 02  0.66977D 01  0.55124D 02  0.27582D 02  0.24573D-01  0.41761D-01
RESPECTIVE ELASTICITIES ARE :
============================

 0.23472D-01  0.75734D-04  0.31374D 01  0.17582D 02  0.59342D 01  0.73257D 01 -0.36624D 01  0.23446D 00 -0.86412D 00
```

```
      DEL P VECTOR
 0.119236980 03   0.694731650-02   0.475904730 04   0.399035490 05   0.499994160 00   0.113362450 01   0.186929130 00
 0.299985610 00

      SECOND ORDER MOVE VECTOR
-0.734700750-01  -0.695577170-07  -0.381124880-01  -0.445775200 00   0.134015910-19   0.421401280-04   0.133869540-02
-0.702828230-08

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X7  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.155400 01 -0.934070-04 -0.628450 02 -0.526950 03 -0.713010 02 -0.597850 03 -0.298920 03 -0.328530 00 -0.466170 00

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X7  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

-0.175010 01 -0.113040-03 -0.706690 02 -0.592560 03 -0.801710 02 -0.672230 03 -0.336110 03 -0.365160 00 -0.525010 00

RESPECTIVE ELASTICITIES ARE :
=============================

 -0.224520 03 -0.725390-02 -0.226650 03 -0.126700 04 -0.428530 02 -0.533490 03  0.269490 03 -0.214710 02  0.673520 02

      DEL P VECTOR
 0.119236980 03   0.694731650-02   0.475904730 04   0.399035490 05   0.499994160 00   0.113362450 01   0.186929130 00
 0.299985610 00

      SECOND ORDER MOVE VECTOR
 0.488348970-05  -0.120745990-10  -0.813179110-05  -0.662254530-04  -0.419223380-14  -0.153694250-09  -0.702828230-08
 0.155032810-07

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X8  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

 0.737790-04  0.434640-08  0.300450-02  0.251940-01  0.340620-02  0.285630-01  0.142810-01  0.140470-04  0.215940-04

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X8  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
-------------------------------------------------------------------------------------------

 0.924230-04  0.961130-08  0.367310-02  0.308010-01  0.416410-02  0.349180-01  0.174590-01  0.172040-04  0.273470-04

RESPECTIVE ELASTICITIES ARE :
=============================

 0.739420-02  0.384490-06  0.734660-02  0.410700-01  0.138810-02  0.174630-01 -0.372970-02  0.650990-03 -0.222790-02
```

EXAMPLE-3 : SOLUTION AND SESITIVITY ANALYSIS OF CORRUGATED BULKHEAD

SYMPUT PARAMETER CARD

N= 6 NN= 45 M= 19 MZ= 0' ITEST= 1 IPR= 0 ITES= 0 NSWIT= 0 IID= 15

TYPE NAME	TYPE #	TYPE DESCRIP.	VAR. NAME	NUMBR. ASSND.	ARG. VARIAB. NAME	NUMBR.	FUNCT. TYPE	NUMBR. ASSNC.	CONSTANT C1	CONSTANT C2
V	ORIG.	VARIABLE	X1	1	0.45800D 02					
V	ORIG.	VARIABLE	X2	2	0.43200D 02					
V	ORIG.	VARIABLE	X3	3	0.30500D 02					
V	ORIG.	VARIABLE	X4	4	0.12000D 01					
V	ORIG.	VARIABLE	X5	5	0.12000D 01					
V	ORIG.	VARIABLE	X6	6	0.13000D 01					
P	SENSITIVITY	PAR	LT					1	0.49500D 03	
P	SENSITIVITY	PAR	LM					2	0.38500D 03	
P	SENSITIVITY	PAR	LB					3	0.31500D 03	
P	SENSITIVITY	PAR	.TTM					4	0.10500D 01	
P	SENSITIVITY	PAR	TMM					5	0.10500D 01	
P	SENSITIVITY	PAR	TBM					6	0.10500D 01	
P	SENSITIVITY	PAR	K2					7	0.15000D 00	
D	DATA	DEFINITION	E						0.800000000 00	
D	DATA	DEFINITION	GB						0.373660000 04	
D	DATA	DEFINITION	KHT						0.345600000-04	
D	DATA	DEFINITION	KHM						0.651000000-04	
D	DATA	DEFINITION	KHB						0.893300000-04	
D	DATA	DEFINITION	CT						0.111770000-01	
D	DATA	DEFINITION	CM						0.137650000-01	
D	DATA	DEFINITION	CB						0.155660000-01	
S	3	SEPARABL	Z1	7	X2	2	POW	10	0.200000000 01	0.0
S	3	SEPARABL	Z1	7	X3	3	POW	11	0.200000000 01	-.100000000 01
S	3	SEPARABL	Z2	8	X1	1	IDN	2	0.0	0.0
S	3	SEPARABL	Z2	8	Z1	7	POW	10	0.500000000 00	0.0
Q	4	QUADRATC	Z3	9	X3	3	LIN	3	0.500000000 00	0.0
Q	4	QUADRATC	Z3	9	X4	4	IDN	2	0.0	0.0
Q	4	QUADRATC	Z4	10	X3	3	LIN	3	0.500000000 00	0.0
Q	4	QUADRATC	Z4	10	X5	5	IDN	2	0.0	0.0
Q	4	QUADRATC	Z5	11	X3	3	LIN	3	0.500000000 00	0.0
Q	4	QUADRATC	Z5	11	X6	6	IDN	2	0.0	0.0
S	3	SEPARABL	Z6	12	X2	2	LIN	3	0.333333330 00	0.0
S	3	SEPARABL	Z6	12	X1	1	LIN	3	0.800000000 00	0.0
Q	4	QUADRATC	Z7	13	Z3	9	IDN	2	0.0	0.0
Q	4	QUADRATC	Z7	13	Z6	12	IDN	2	0.0	0.0
Q	4	QUADRATC	Z8	14	Z4	10	IDN	2	0.0	0.0
Q	4	QUADRATC	Z8	14	Z6	12	IDN	2	0.0	0.0
Q	4	QUADRATC	Z9	15	Z5	11	IDN	2	0.0	0.0
Q	4	QUADRATC	Z9	15	Z6	12	IDN	2	0.0	0.0
S	3	SEPARABL	Z10	16	Z2	8	LIN	3	0.345600000-04	0.0
S	3	SEPARABL	Z11	17	Z2	8	LIN	3	0.651000000-04	0.0
S	3	SEPARABL	Z12	18	Z2	8	LIN	3	0.893800000-04	0.0
S	3	SEPARABL	Z13	19	Z10	16	LIN	3	0.495000000 03	0.0
									LT	
S	3	SEPARABL	Z14	20	Z11	17	LIN	3	0.385000000 03	0.0
									LM	
S	3	SEPARABL	Z15	21	Z12	18	LIN	3	0.315000000 03	0.0

```
S   3   SEPARABL   Z16   22    Z13   19    LIN    3 0.495000000 03  0.0
                                                    LT
S   3   SEPARABL   Z17   23    Z14   20    LIN    3 0.385000000 03  0.0
                                                    LM
S   3   SEPARABL   Z18   24    Z15   21    LIN    3 0.315000000 03  0.0
                                                    L3
S   3   SEPARABL   G1    25    X2    2     IDN    2 0.0             0.0
S   3   SEPARABL   G1    25    X3    3     LIN    3 -.100000000 01  0.0
S   3   SEPARABL   G2    26    Z7    13    IDN    2 0.0             0.0
S   3   SEPARABL   G2    26    Z16   22    LIN    3 -.100000000 01  0.0
S   3   SEPARABL   G3    27    Z8    14    IDN    2 0.0             0.0
S   3   SEPARABL   G3    27    Z17   23    LIN    3 -.100000000 01  0.0
S   3   SEPARABL   G4    28    Z9    15    IDN    2 0.0             0.0
S   3   SEPARABL   G4    28    Z18   24    LIN    3 -.100000000 01  0.0
Q   4   QUADRATC   G5    29    Z7    13    IDN    2 0.0             0.0
Q   4   QUADRATC   G5    29    X3    3     LIN    3 0.500000000 00  0.0
S   3   SEPARABL   G5    29    Z16   22    POW   11 0.133333330 01  -.220000000 01
Q   4   QUADRATC   G6    30    Z8    14    IDN    2 0.0             0.0
Q   4   QUADRATC   G6    30    X3    3     LIN    3 0.500000000 00  0.0
S   3   SEPARABL   G6    30    Z17   23    POW   11 0.133333330 01  -.220000000 01
Q   4   QUADRATC   G7    31    Z9    15    IDN    2 0.0             0.0
Q   4   QUADRATC   G7    31    X3    3     LIN    3 0.500000000 00  0.0
S   3   SEPARABL   G7    31    Z18   24    POW   11 0.133333330 01  -.220000000 01
S   3   SEPARABL   G8    32    X4    4     LIN    4 -.100000000 01  0.105000000 01
                                                                    TTM
S   3   SEPARABL   G9    33    X4    4     LIN    3 -.100000000 01  0.0
S   3   SEPARABL   G9    33    X1    1     LIN    4 0.111770000-01  0.150000000 00
                                                                    K2
S   3   SEPARABL   G10   34    X4    4     LIN    3 -.100000000 01  0.0
S   3   SEPARABL   G10   34    X2    2     LIN    4 0.111770000-01  0.150000000 00
                                                                    K2
S   3   SEPARABL   G11   35    X5    5     LIN    4 -.100000000 01  0.105000000 01
                                                                    TMM
S   3   SEPARABL   G12   36    X5    5     LIN    3 -.100000000 01  0.0
S   3   SEPARABL   G12   36    X1    1     LIN    4 0.137650000-01  0.150000000 00
                                                                    K2
S   3   SEPARABL   G13   37    X5    5     LIN    3 -.100000000 01  0.0
S   3   SEPARABL   G13   37    X2    2     LIN    4 0.137650000-01  0.150000000 00
                                                                    K2
S   3   SEPARABL   G14   38    X6    6     LIN    4 -.100000000 01  0.105000000 01
                                                                    TBM
S   3   SEPARABL   G15   39    X6    6     LIN    3 -.100000000 01  0.0
S   3   SEPARABL   G15   39    X1    1     LIN    4 0.155660000-01  0.150000000 00
                                                                    K2
S   3   SEPARABL   G16   40    X6    6     LIN    3 -.100000000 01  0.0
S   3   SEPARABL   G16   40    X2    2     LIN    4 0.155660000-01  0.150000000 00
                                                                    K2
S   3   SEPARABL   Z19   41    X1    1     IDN    2 0.0             0.0
S   3   SEPARABL   Z19   41    X2    2     IDN    2 0.0             0.0
S   3   SEPARABL   Z20   42    Z19   41    LIN    3 0.375660000 04  0.0
S   3   SEPARABL   Z21   43    X4    4     LIN    3 0.495000000 03  0.0
                                                    LT
S   3   SEPARABL   Z21   43    X5    5     LIN    3 0.385000000 03  0.0
                                                    LM
S   3   SEPARABL   Z21   43    X6    6     LIN    3 0.315000000 03  0.0
                                                    LB
Q   4   QUADRATC   Z22   44    Z20   42    IDN    2 0.0             0.0
Q   4   QUADRATC   Z22   44    Z21   43    IDN    2 0.0             0.0
Q   4   QUADRATC   OBJ   45    Z22   44    IDN    2 0.0             0.0
Q   4   QUADRATC   OBJ   45    Z2    n     POW   10 -.100000000 01  0.0

ARRAY OF CONSTANTS CC(I.)
*******************

LOCATION      VALUE
```

```
NONLINEAR PROGRAMMING ROUTINE-SUMT VERSION 4 08/10/71

   N=  6       M= 19       MZ=  0

     MAX. TIME= 0.30000000D 03    K= 0.10000000D 03    RATIO= 0.40000000D 01      EPSILON= 0.10000000D-04    THETA= 0.10000000D-05

OPTIONS SELECTED
   3      2      2      1      1      1      1      2      1      1

TOLERANCES
     0.10000000D-02        0.10000000D-02

SECOND SET OF OPTIONS
   1      1
     F=  0.63796000D 07      P=  0.0              G=  0.0                   RSIGMA=  0.0                   H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------------
  X1=  0.45800000D 02     X2=  0.43200000D 02     X3=  0.30500000D 02     X4=  0.12000000D 01     X5=  0.12000000D 01
  X6=  0.13000000D 01
   THE CONSTRAINT VALUES
    0.45800000D 02        0.43200000D 02        0.30500000D 02        0.12700000D 02        0.23712310D 03        0.19687260D 03
    0.33435060D 03        0.19352510D 04       -0.40602190D 03        0.23396720D 04        0.15000000D 00        0.53809340D 00
    0.56715360D 00        0.15000000D 00        0.41956300D 00        0.45535210D 00        0.25000000D 00        0.43707720D 00
    0.47754880D 00

    ORTHOGONAL MOVE

    DEL P VECTOR
-0.31385337D 02  -0.29407003D 03    0.134327320D 04    0.191527300D 04    0.135899730D 05    0.157832840D 04

    SECOND ORDER MOVE VECTOR
 0.79081510D 01    0.198890440D 01    0.100000000D 01    0.0              0.0              0.0

**************************************************
     POINT=    1         DOTI=  0.51019610D 03      RHO=  0.10000000D 07       MAGNITUDE=  0.13387040D 05       PHASE= 1
   F=  0.0              P= -0.41387020D 04     G= -0.19000000D 04     RSIGMA= -0.41387020D 04      H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------------
  X1=  0.53708151D 02     X2=  0.45188904D 02     X3=  0.31500000D 02     X4=  0.12000000D 01     X5=  0.12000000D 01
  X6=  0.13000000D 01
   THE CONSTRAINT VALUES
    0.53708150D 02        0.45188900D 02        0.31500000D 02        0.13688900D 02        0.36753450D 03        0.26385720D 03
    0.42448010D 03        0.28351640D 04        0.88734440D 02        0.33566560D 04        0.15000000D 00        0.44970400D 00
    0.54492360D 00        0.15000000D 00        0.31070730D 00        0.42797470D 00        0.25000000D 00        0.31397890D 00
    0.44658950D 00

*****THE FEASIBLE STARTING POINT TO BE USED IS ...
   F=  0.62892580D 07      P= -0.41387020D 04      G=  0.0                  RSIGMA= -0.41387020D 04      H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
---------------------------------------------------------------
  X1=  0.53708151D 02     X2=  0.45188904D 02     X3=  0.31500000D 02     X4=  0.12000000D 01     X5=  0.12000000D 01
  X6=  0.13000000D 01
   THE CONSTRAINT VALUES
    0.53708150D 02        0.45188900D 02        0.31500000D 02        0.13688900D 02        0.36753450D 03        0.26385720D 03
    0.42448010D 03        0.28351640D 04        0.88734440D 02        0.33566560D 04        0.15000000D 00        0.44970400D 00
    0.54492360D 00        0.15000000D 00        0.31070730D 00        0.42797470D 00        0.25000000D 00        0.31397890D 00
    0.44658950D 00

    ORTHOGONAL MOVE

    DEL P VECTOR
 0.94097440D 04    0.57747340D 05   -0.60335750D 05   -0.21224800D 07   -0.15744970D 07   -0.13532490D 07
```

```
-0.90538071D 01  -0.23163693D 01  -0.35292702D 00  -0.10000000D 01   0.0             0.0

*********************************************
      POINT=    1        DOTT=  0.15733867D 07       RHO=  0.10000000D 03       MAGNITUDE=  0.30005534D 07       PHASE= 2
    F=  0.59924760 07     P=  0.59893320 07     G=  0.59915750 07      RSIGMA= -0.31432330 04      H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
  X1=  0.52350553D 02     X2=  0.44841570D 02     X3=  0.31447079D 02     X4=  0.10500523D 01     X5=  0.12000000D 01
  X6=  0.13000000D 01
   THE CONSTRAINT VALUES
     0.52350550 02        0.44841570 02        0.31447080 02        0.13394440 02        0.22425390 03        0.25862630 03
     0.41380710 03        0.71309830 03        0.14928380 03        0.33321100 04        0.52280670-04        0.31493010 00
     0.39885810 00        0.15000000 00        0.32939460 00        0.43275580 00        0.25000000 00        0.33511130 00
     0.45199610 00

     ORTHOGONAL MOVE

     DEL P VECTOR
 0.93910531D 04    0.37913339D 05  -0.68779327D 05  -0.21638425D 06  -0.16473062D 07  -0.13552045D 07

     SECOND ORDER MOVE VECTOR
 0.65330939D 01    0.18812360D 02    0.16147683D 02  -0.44514847D-05  -0.10000000D 01    0.0

*********************************************
      POINT=    2        DOTT=  0.13093910 07       RHO=  0.10000000 03       MAGNITUDE=  0.21455210 07       PHASE= 2
    F=  0.5793322D 07     P=  0.5790785D 07     G=  0.5791422D 07      RSIGMA= -0.25374070 04      H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
  X1=  0.53330011D 02     X2=  0.47661965D 02     X3=  0.33867980D 02     X4=  0.10500516D 01     X5=  0.10500775D 01
  X6=  0.13000000D 01
   THE CONSTRAINT VALUES
     0.53330010 02        0.47661970 02        0.33867980 02        0.13793990 02        0.30555240 03        0.20295670 03
     0.51857400 03        0.30223350 04        0.24410730 03        0.62902370 04        0.51613290-04        0.30398210 00
     0.36733380 00        0.77531280-04        0.16598590 00        0.24401060 00        0.25000000 00        0.31986510 00
     0.40809380 00

     ORTHOGONAL MOVE

     DEL P VECTOR
 0.93106249D 04    0.37288030D 05  -0.66713736D 05  -0.21144592D 06  -0.37436219D 06  -0.13670286D 07

     SECOND ORDER MOVE VECTOR
-0.44541843D 01    0.17454048D 01    0.24181495D 01  -0.13894792D-05  -0.71351486D-05  -0.10000000D 01

*********************************************
      POINT=    3        DOTT=  0.12293190 07       RHO=  0.10000000 03       MAGNITUDE=  0.14351140 07       PHASE= 2
    F=  0.5484619D 07     P=  0.5483084D 07     G=  0.5482719D 07      RSIGMA= -0.15349420 04      H=  0.0
   THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
------------------------------------------------------------------
  X1=  0.52216326D 02     X2=  0.48098175D 02     X3=  0.34472321D 02     X4=  0.10500513D 01     X5=  0.10500757D 01
  X6=  0.10500812D 01
   THE CONSTRAINT VALUES
     0.52216830 02        0.48098170 02        0.34472320 02        0.13625950 02        0.32001090 03        0.21572070 03
     0.28567980 03        0.36721970 04        0.94103330 03        0.27596710 04        0.51266030-04        0.31642380 00
     0.36245800 00        0.75745070-04        0.18131110 00        0.23800440 00        0.81164640-04        0.67274050-01
     0.15138500 00

     ORTHOGONAL MOVE

     DEL P VECTOR
 0.92581564D 04    0.36971025D 05  -0.65465086D 05  -0.21156389D 06  -0.33394754D 06  -0.14146591D 06

     SECOND ORDER MOVE VECTOR
-0.77821637D 00  -0.12036125D 01  -0.10700000D 01    0.0             0.0             0.0
```

```
*************************************************
     POINT=  33       DJIT=  0.1187357D-02      RHO=  0.9765625D-01      MAGNITUDE=  0.1389334D 06      PHASE= 2
   F=  0.5248023D 07      P=  0.5248327D 07      G=  0.5248021D 07      KSIGMA=  0.4257422D 01      H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
  X1=  0.5781833 4D 02    X2=  0.5781834 5D 02     X3=  0.3568833 2D 02     X4=  0.1050000 0D 01     X5=  0.1050010 01D 01
  X6=  0.1050000 50D 01
    THE CONSTRAINT VALUES
     0.5781833D 02       0.5781833D 02       0.3568833D 02       0.2213000D 02       0.3529274D 03       0.2308512D 03
     0.3115359D 03       0.3501215D 04       0.3591984D-02       0.2330933D 04       0.4770149D-07       0.2537645D 00
     0.2537644D 00       0.4526797D-07       0.1041307D 00       0.1041300D 00       0.4887491D-06       0.3077094D-06
     0.1243014D-06
APPARENTLY ROUNDOFF ERRORS PREVENT A MORE ACCURATE DETERMINATION OF THE MINIMUM OF THIS SUBPROBLEM.

   2ND  ORDER ESTIMATES
     F=  0.5248022D 07      P=  0.5248023D 07      G=  0.5248022D 07      RSIGMA=  0.0                    H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
  X1=  0.5781832 2D 02    X2=  0.5781832 3D 02     X3=  0.3568832 97D 02    X4=  0.1050000 00D 01    X5=  0.1050000 00D 01
  X6=  0.1050000 00D 01
    THE CONSTRAINT VALUES
     0.5781832D 02       0.5781832D 02       0.3568833D 02       0.2213003D 02       0.3529272D 03       0.2308840D 03
     0.3115352D 03       0.3501210D 04       0.8764775D-04       0.2330920D 04       0.1154691D-08       0.2537646D 00
     0.2537646D 00       0.2325524D-08       0.1041308D 00       0.1041308D 00       0.1195437D-07       0.7511207D-08
     0.3033892D-08

   1ST  ORDER ESTIMATES
     F=  0.5248022D 07      P=  0.5248023D 07      G=  0.5248022D 07      RSIGMA=  0.0                    H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
  X1=  0.5781832 2D 02    X2=  0.5781832 2D 02     X3=  0.3568829 7D 02     X4=  0.1050000 00D 01    X5=  0.1050000 00D 01
  X6=  0.1050000 00D 01
    THE CONSTRAINT VALUES
     0.5781832D 02       0.5781832D 02       0.3568830D 02       0.2213003D 02       0.3529272D 03       0.2308809D 03
     0.3115352D 03       0.3501210D 04       0.7764443D-04       0.2330920D 04       0.1032981D-08       0.2537646D 00
     0.2537646D 00       0.2064054D-08       0.1041308D 00       0.1041303D 00       0.1032740D-07       0.6658967D-08
     0.2689538D-08

  LAGRANGE MULTIPLIERS
     F=  0.5248022D 07      P=  0.5248023D 07      G=  0.5248022D 07      RSIGMA=  0.0                    H=  0.0
    THE CURRENT VALUE OF THE PROB-M(I.E.X(I) ) VARS.ARE;
-----------------------------------------------------------------
  X1=  0.5781832 2D 02    X2=  0.5781832 2D 02     X3=  0.3568829 7D 02     X4=  0.1050000 00D 01    X5=  0.1050000 00D 01
  X6=  0.1050000 00D 01
    THE CONSTRAINT VALUES
     0.1689019D-02       0.1689019D-02       0.2730366D-02       0.4412433D-02       0.2757035D-03       0.4229719D-03
     0.3134671D-03       0.2789211D-04       0.2718728D 02       0.4149576D-04       0.2047260D 07       0.3848302D 00
     0.3848304D 00       0.1025969D 07       0.9378235D 00       0.9378250D 00       0.1773049D 06       0.3173652D 06
     0.7856437D 06

SENSITIVITY OF SOLUTION VECTOR WRT PROBLEM PARAMETERS
*************************************************

PROBLEM PARAMETERS ARE
======================

  LF            LM            LB            IT4           I44           I 4           K2

   DEL-P VECTOR
  0.5781833 4D 02     0.5781834 5D 02     0.3568833 2D 02     0.1050000 0D 01     0.1050010 0D 01     0.1050000 5D 01
```

```
 0.12570602D-07   0.90024974D-08   0.16723955D-08   0.14272375D-17  -0.92927837D-17   0.14013293D-09

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

 0.14572D-06  0.12483D-05 -0.44038D-06  0.61256D-04 -0.59948D-04  0.57289D 02 -0.57284D 02
THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X1  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.25549D-07 -0.37638D-06  0.77200D-07 -0.25350D-04  0.48653D-04  0.54054D 02 -0.54061D 02
RESPECTIVE ELASTICITIES ARE :
=============================

 -0.21874D-06 -0.25062D-05  0.42059D-06 -0.47852D-06  0.88355D-06  0.98164D 00 -0.14025D 00

     DEL P VECTOR
  0.57818334D 02   0.57818346D 02   0.35688303D 02   0.10500000D 01   0.10500001D 01   0.10500005D 01

     SECOND ORDER MOVE VECTOR
  0.90024974D-08   0.95847473D-08   0.17343580D-08   0.13401549D-17  -0.88193831D-17   0.14013298D-09

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

 0.13683D-06  0.12284D-05 -0.44927D-06  0.57514D-04 -0.94895D-04  0.57289D 02 -0.57283D 02
THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X2  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.23994D-07 -0.35504D-06  0.78757D-07 -0.24744D-04  0.46425D-04  0.54054D 02 -0.54061D 02
RESPECTIVE ELASTICITIES ARE :
=============================

 -0.20542D-06 -0.23641D-05  0.42907D-06 -0.44936D-06  0.84310D-06  0.98164D 00 -0.14025D 00

     DEL P VECTOR
  0.57818334D 02   0.57818346D 02   0.35688303D 02   0.10500000D 01   0.10500001D 01   0.10500005D 01

     SECOND ORDER MOVE VECTOR
  0.16723955D-08   0.17343580D-08   0.33752372D-09   0.22035520D-13  -0.11923508D-11   0.25446407D-10

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X3  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

 0.22498D-07  0.93342D-01 -0.83932D-07  0.94575D-05 -0.12835D 02  0.10403D 02 -0.10402D 02
THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X3  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
------------------------------------------------------------------------------------------

-0.39469D-08  0.88092D-01  0.14711D-07 -0.41693D-05 -0.17112D 02  0.98157D 01 -0.98173D 01
RESPECTIVE ELASTICITIES ARE :
=============================

 -0.54743D-07  0.95333D 00  0.129[illegible]D-05 -0.115[illegible]D-06 -0.35[illegible] [illegible]  0.2[illegible] 00 -[illegible]

     DEL P VECTOR
  0.57818334D 02   0.57818346D 02   0.35688303D 02   0.10500000D 01   0.10500001D 01   0.10500005D 01
```

```
0.14272373D-17   0.13431347D-17   0.22035520D-15   0.20775440D-15   0.53705037D-21   0.17742184D-19

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X4  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
--------------------------------------------------------------------------------------------
-0.86695D-10 -0.30499D-10 -0.62774D-16  0.87169D 00  0.41526D-08  0.81527D-08 -0.81519D-08

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X4  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
--------------------------------------------------------------------------------------------
 0.15202D-10  0.13095D-10  0.34483D-15  0.84149D 00 -0.18021D-08 -0.35054D-08  0.35012D-08

RESPECTIVE ELASTICITIES ARE :
=============================
  0.71668D-03  0.48015D-08  0.13345D-12  0.84149D 00 -0.18021D-08 -0.35054D-08  0.53014D-09

     DEL P VECTOR
  0.57818334D 02   0.57818346D 02   0.35688333D 02   0.10500000D 01   0.10500001D 01   0.10500005D 01

     SECOND ORDER MOVE VECTOR
 -0.92927837D-17  -0.88193831D-17  -0.11923508D-11   0.34965007D-21   0.32372780D-13  -0.12965933D-18

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X5  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
--------------------------------------------------------------------------------------------
 0.39781D-10 -0.43609D-10  0.39773D-10  0.16724D-07  0.89176D 00 -0.55007D-07  0.63646D-07

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X5  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
--------------------------------------------------------------------------------------------
-0.69543D-11 -0.12030D-10 -0.69390D-11 -0.71379D-08  0.84149D 00  0.25958D-07 -0.30487D-07

RESPECTIVE ELASTICITIES ARE :
=============================
 -0.32785D-08 -0.44108D-08 -0.20817D-08 -0.71879D-08  0.84148D 00  0.25958D-07 -0.43553D-08

     DEL P VECTOR
  0.57818334D 02   0.57818346D 02   0.35688333D 02   0.10500000D 01   0.10500001D 01   0.10500005D 01

     SECOND ORDER MOVE VECTOR
  0.14013293D-09   0.14013298D-09   0.25446837D-10   0.19942184D-19  -0.12965933D-18   0.21813108D-11

THE ESTIMATES OF ZEROTH ORDER SENSITIVITY OF  X6  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
--------------------------------------------------------------------------------------------
 0.20361D-08  0.18151D-07 -0.70872D-08  0.85591D-05 -0.13951D-05  0.89176D 00  0.64178D-05

THE ESTIMATES OF FIRST  ORDER SENSITIVITY OF  X6  WRT PROBLEM PARAMETERS ARE RESPECTIVELY :
--------------------------------------------------------------------------------------------
-0.35710D-09 -0.52343D-08  0.12423D-08 -0.36823D-06  0.63291D-06  0.84141D 00 -0.29211D-05

RESPECTIVE ELASTICITIES ARE :
=============================
 -0.16435D-16 -0.19151D-05  0.57213D-06 -0.36822D-06  0.63291D-06  0.84141D 00 -0.44172D-06
```

APPENDIX 2

"Sensitivity Analysis in Nonlinear Programming Using Factorable Symbolic Input"
(borrowed directly from [4])

THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
Institute for Management Science and Engineering

SENSITIVITY ANALYSIS IN NONLINEAR PROGRAMMING USING FACTORABLE SYMBOLIC INPUT

by

Anura de Silva
Garth P. McCormick

1. Introduction

This paper develops a methodology to utilize the input to nonlinear programs in symbolic factorable form to perform first order sensitivity analysis on the solution vector $x(y) \varepsilon E^n$ of a general nonlinear program, where $y \varepsilon E^k$ is the parametric vector. In Section 2 we discuss the key results of sensitivity analysis theory in nonlinear programming. In Section 3 we develop the formulae underlying the new methodology to utilize the symbolic factorable input to perform sensitivity analysis on the solution vector $x(y)$.

The above are implemented via the SUMT algorithm [4] for nonlinear programming which uses a penalty function technique. Section 4 briefly discusses the penalty function-based implementation of the sensitivity analysis methodology. An extrapolation result based on the penalty function parameter r is stated and incorporated in the computer code.

2. Relevant Results from NLP Sensitivity Theory

Consider the problem

$$\left.\begin{aligned} &\underset{x\varepsilon E^n}{\text{MIN}} \quad F(x,y) \\ &\text{s.t.} \quad G(x,y) \geqq 0 \\ &\qquad\;\; H(x,y) = 0 \\ &\text{where } G : E^n \times E^k \to E^m \\ &\qquad\;\; H : E^n \times E^k \to E^p \end{aligned}\right\} \text{Problem } M(y)$$

By sensitivity information we mean the rates of change of the (locally) optimal objective function value and the corresponding optimizing vector with respect to changes in the parametric vector. If $x(y)$ is an n-vector and y a k-vector, the first order sensitivity information for Problem M(y) is:

(a) The rate of change of the solution value with respect to the parametric vector, $D_y F[x(y),y] \; \varepsilon E^k$.

(b) The rate of change of the optimizing vector with respect to the parametric vector, $\frac{dx(y)}{dy}$, where

$$\frac{dx(y)}{dy} = \begin{bmatrix} \frac{\partial x_1(y)}{\partial y_1} & \cdot\;\cdot & \frac{\partial x_1(y)}{\partial y_k} \\ \vdots & & \vdots \\ \frac{\partial x_n(y)}{\partial y_1} & \cdot\;\cdot & \frac{\partial x_n(y)}{\partial y_k} \end{bmatrix} \tag{1}$$

The first order sensitivity results for the general nonlinear programming problem were first stated by Fiacco and McCormick in [3], Theorem 6, where they give conditions under which a smooth (once continuously differentiable, abbreviated OCD) trajectory of the optimizing Kuhn-Tucker triple $[x(y),u(y),w(y)]$ exist for y close to an initial value y_0 . The u and w are the Lagrange multiplier vectors associated with the inequality

constraints G and equality constraints H of Problem $M(y)$, in the Lagrangian,

$$L(x,y,u,w) = F(x,y) - \sum_{i=1}^{m} u_i \, G_i(x,y) + \sum_{j=1}^{p} w_j \, H_j(x,y) \qquad (2)$$

where $u_i, G_i; i = 1,\ldots,m$ and $w_j, H_j : j = 1,\ldots,p$ are the components of the mappings G and H and the Lagrange multipliers u and w. Here, $u(y)$ and $w(y)$ are the values of these multipliers associated with the optimal value $x(y)$. They assume the following to hold; (a) twice continuous differentiability (abbreviated TCD) of $L(x,y,u,w)$ in (x,y) in a neighborhood of $(x(y_0),y_0)$, (b) second order sufficiency conditions (abbrev. SOSC) at $(x(y_0),y_0)$, (c) linear independence of binding constraints (abbrev. LIB) at $(x(y_0),y_0)$, (d) strict complementary slackness (abbrev. SCS). Here, (c) and (d) are regularity conditions they invoke to obtain the key result in a useful form. In a more recent paper Fiacco [2] obtains a similar result for somewhat more general parametric problem, such as $M(y)$.

The crux of the theory is the application of the implicit function theorem (see [6] for a general treatment of the implicit function theorem) to the first order Kuhn-Tucker system of $M(y)$, which is

$$\begin{aligned} \nabla_x L(x,y,u,w) &= 0 \\ u_i \, G_i(x,y) &= 0 \, , \; i = 1,\ldots,m \\ H_j(x,y) &= 0 \, , \; j = 1,\ldots,p \, . \end{aligned} \qquad (3)$$

Consider a system of equations

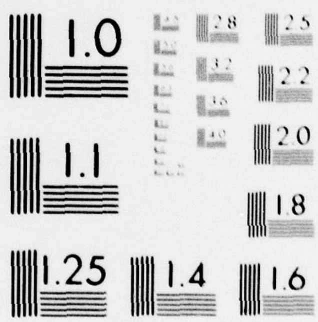
$$D(x,y) = 0 \, , \; x \in E^n \text{ and } y \in E^k \qquad \text{Problem } P(y) \, ,$$

where $D : E^n \times E^k \to E^\ell$, ℓ being the number of scalar valued functions in D, which are assumed OCD. This ensures the existence of the Jacobians of $D(x,y)$ with respect to x and y, denoted J_xD and J_yD, respectively. With the sensitivity analysis context in view, the implicit function theorem could be stated as follows:

AD-A074 903 GEORGE WASHINGTON UNIV WASHINGTON D C INST FOR MANAG--ETC F/G 12/1
SYMBOLIC FACTORABLE SUMT: WHAT IS IT, HOW IS IT USED,(U)
MAY 79 A GHAEMI, G P MCCORMICK N00014-75-C-0611
UNCLASSIFIED SERIAL-T-402 NL

2 OF 2
AD
A074903

END
DATE
FILMED
11-79
DDC



MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

Theorem 1. Suppose $D : E^n \times E^k \to E^\ell$; $(x(y_0),y_0)$ is a zero of $D(x,y)$; $D(x,y)$ is OCD near $(x(y_0),y_0)$; J_xD is nonsingular at $(x(y_0),y_0)$.

Then, there exist open sets $S_x(x(y_0)) \subset E^n$ and $S_y(y_0) \subset E^k$ such that for any $y \in \bar{S}_y(y_0)$ there exists a unique solution for x in terms of y , i.e., $x = x(y) \in S_x(x(y_0))$, which is continuous in y . Moreover $x(y)$ is OCD at y_0 and

$$\nabla_y x(y_0) = - \left[J_x D_0\right]^{-1} \left[J_y D_0\right] ,$$

where the subscript 0 on D denotes evaluation at $(x(y_0),y_0)$.

If $P(y)$ is compared with (3) the role of Theorem 1 in the context of sensitivity analysis of the solution vector becomes evident. The assumed TCD of the problem functions gives the necessary OCD for the system in (3). The additional conditions of linear independence of the binding constraint gradients and SCS were assumed by Fiacco and McCormick [3] and Fiacco [2] to produce desirable results such as the uniqueness of the Lagrange multipliers $[u(y),w(y)]$ associated with the optimal $x(y)$ and the invariance of the set of inequality constraints binding at $x(y_0)$ in a neighborhood of y_0 . The following representation for the Kuhn-Tucker triple,

$$X \triangleq (x,u,w)$$

$$\text{and} \quad X(y) \triangleq [x(y),u(y),w(y)]$$

will be adopted for notational convenience. For the same reason, the first order Kuhn-Tucker system of (3) will be denoted as follows.

$$R(x,u,w,y) = 0 \quad \text{or} \quad \bar{R}(X,y) = 0 . \tag{3b}$$

Differentiating (3b) partially with respect to X yields the Jacobian of (3b) or (3) with respect to X,

$$J_X\bar{R}(X,y) = \left[\begin{array}{c|ccc|ccc}
\nabla^2_{xx}L & -\nabla_x G_1, & \cdots & -\nabla_x G_m & \nabla_x H_1, & \cdots & \nabla_x H_p \\
\hline
u_1\nabla_x^T G_1 & G_1 & & 0 & & & \\
\vdots & & \ddots & & & 0 & \\
u_m\nabla_x^T G_m & 0 & & G_m & & & \\
\hline
\nabla_x^T H_1 & & & & & & \\
\vdots & & 0 & & & 0 & \\
\nabla_x^T H_p & & & & & &
\end{array}\right] \tag{4}$$

Differentiating (3b) partially with respect to y yields the Jacobian with respect to this parametric vector,

$$J_y\bar{R}(X,y) = \left[\nabla^2_{yx}L^T,\ u_1\nabla_y G_1, \ldots, u_m\nabla_y G_m,\ \nabla_y H_1, \ldots, \nabla_y H_p\right]^T . \tag{5}$$

At a Kuhn-Tucker point, (3b) could be written as

$$\bar{R}[X(y),y] = 0 . \tag{3c}$$

Totally differentiating (3c) with respect to y, the following relationship is obtained.

$$J_X\bar{R}(X(y),y)\,\frac{dX(y)}{dy} + J_y\bar{R}(X(y),y) = 0 . \tag{6}$$

The principal result immediately follows.

$$\frac{dX(y)}{dy} = -\,J_X\bar{R}(X(y),y)^{-1} \cdot J_y\bar{R}(X(y),y) . \tag{7}$$

Equation (7) is almost the same as the result stated under Theorem 1 but is stated with respect to the entire Kuhn-Tucker triple, X . Note that (7) is stated for y in a neighborhood of y_0 , whereas the result of Theorem 1 was only at y_0 . This is possible due to the slightly stronger assumptions made by Fiacco and McCormick [3] and Fiacco [2].

The above result is the basis for the development of a methodology for sensitivity analysis in nonlinear programming. The computational technique used here is the penalty function method. The same result and technique were used by Armacost and Mylander [1] to compute (1), the principal difference in their methodology being that they approximate $J_y\bar{R}$ by central differencing techniques, and compute all other derivatives manually. The relevant formulae for the penalty function implementation using the logarithmic barrier-quadratic loss penalty function of the SUMT algorithm [4] are discussed in Section 3.

3. Sensitivity Analysis With Symbolic Input

The factorable input technique and computer code to compute gradient vectors and Hessian matrices of problem functions was developed by McCormick [5] to facilitate the use of nonlinear programming computer codes. The code itself has since been extended by de Silva and McCormick to accept input in a symbolic format. This symbolic code has been used to solve several general nonlinear programming problems and the ease with which a problem input can be supplied or modifications done is very encouraging. A sensitivity analysis option turned out to be a natural extension to the symbolic computer code.

A reduced version of the matrix in (1) which provides substantial computational economy, results from the formulae which are developed below; a brief discussion of the factorable input philosophy precedes this development.

In the factorable input method, the problem functions are built up recursively as sums, or sums of pairwise products, of simple transformations of previously defined functions. Initially, these simple transformations

are applied to the original problem variables x_i, $i = 1,\ldots,n$, and the recursive application of the above procedure (shown mathematically in Equation (8)) generates new functions X^i, $i = n+1,\ldots,N$.

McCormick [5] gives a detailed exposition illustrated by an example. He sets $X^i = x_i$, $i = 1,\ldots,n$ and refers to the entire set of X^i, $i = 1,\ldots,N$ as concomitant variable functions (abbrev. CVFs). The general equation for the creation of a new CVF is:

$$X^i(x) = \sum_{p=1}^{i-1} T_p^i[X^p(x)] + \sum_{p=1}^{i-1} \sum_{q=1}^{p} V_{q,p}^i[X^p(x)] \cdot U_{p,q}^i[X^q(x)] \; . \tag{8}$$

In the presence of a parametric vector y, the equation becomes

$$X^i(x,y) = \sum_{p=1}^{i-1} T_p^i[X^p(x,y),y] + \sum_{p=1}^{i-1} \sum_{q=1}^{p} V_{q,p}^i[X^p(x,y),y] \cdot U_{p,q}^i[X^q(x,y),y] \; . \tag{9}$$

The terms in the first summation of (8) or (9) are referred to as <u>separable</u> terms comprising the new CVF $X^i(\cdot)$ and the terms of the second summation of pairwise products are called <u>quadratic</u> terms. A point of note in (9) is that the arguments of the simple transformation T_p^i, $V_{q,p}^i$ or $U_{p,q}^i$ contain y directly as well as indirectly via the argument CVF. The direct presence of the parametric vector y is restricted to, at most, two components because all the simple transformations used in the methodology and code contain, at most, two constant terms. The parametric components are a subset of these constants.

An examination of (7) suggests that second partial derivatives of the problems functions with respect to x and the cross second partial derivatives with respect to x and y are required. This is because $\bar{R}$ already contains first partial derivatives with respect to x. What is developed below will yield the matrix in (1) reduced by postmultiplying it

with a known column vector $\beta \in E^k$ or premultiplying it with a known row vector $\sigma \in E^n$. The postmultiplication produces an n-vector, each component being a weighted sum of the k partial derivatives of a particular $x_i(y)$, with respect to the components of the parametric vector. The weights are the components of $\beta \in E^k$. The ith component of this resulting n-vector can be looked upon as the "total sensitivity" of the variable $x_i(y)$ $(i=1,\ldots,n)$ with respect to all the y_j , $j = 1,\ldots k$, weighted according to β_j , $j = 1,\ldots,k$. The rth component of the n-vector is $\sum_{j=1}^{k} \frac{x_r(y)}{\partial y_j} \beta_j$. The premultiplication produces a k-component vector, the jth component being the "total sensitivity" of a composite of all n variables $x_i(y)$, $i = 1,\ldots,n$, weighted according to σ_i , $i = 1,\ldots,n$, with respect to $y_j(j=1,\ldots,k)$. The ℓth component of this k-vector would be $\sum_{i=1}^{n} \frac{\partial x_i}{\partial y_\ell} \sigma_i$.

For clarity of exposition, (9) will be simplified below to a separable portion of one term or a quadratic term of one product.

Consider the separable case first. Suppose

$$X^i(x,y) = T[X^p(x,y),y] \ . \tag{10}$$

Letting

$$\dot{T} \triangleq \dot{T}\,[X^p(x,y),y] \triangleq \frac{\partial T[X^p(x,y),y]}{\partial X^p} \ , \tag{11}$$

we have

$$\nabla_x X^i(x,y) = \nabla_x X^p(x,y) \cdot \frac{\partial T[X^p(x,y),y]}{\partial X^p} \ . \tag{12a}$$

$$= \dot{T} \cdot \nabla_x X^p(x,y) \ . \tag{12b}$$

In (12b) $\dot{T}$ is readily available since all transformations are simple ones, and $\nabla_x X^p(x,y)$ is built up inductively from its argument CVF.

The expressions for $\nabla^2_{xx} X^i(x,y)$ are already developed and coded in [5], and will not be repeated here. Note that

$$\nabla^2_{yx} X^i(x,y) = \nabla_y \nabla_x X^i(x,y) \tag{13a}$$

$$= D_y D_x \, T[X^p(x,y),y] \; . \tag{13b}$$

In (13b) D denotes total differentiation, although in the previous step the partial differentiation ∇ was used. This is due to the fact that the partial differentiation in (13a) would capture the total variation due to x and y of the expression in (12a), but not so in (13b). Continuing the differentiation with respect to y of (12b), as in (13b), we get

$$\nabla^2_{yx} X^i(x,y) = \nabla^2_{yx} X^p(x,y) \cdot \dot{T} + \nabla_x X^p(x,y) \left\{ \ddot{T}\,[X^p(x,y),y]\; \nabla^T_y X^p(x,y) + \nabla^T_y \dot{T} \right\} . \tag{14}$$

In the above, $\ddot{T} \triangleq \ddot{T}\,[X^p(x,y),y] \triangleq \dfrac{\partial^2 T[X^p(x,y),y]}{\partial (X^p)^2}$. (15)

In (14), $\dot{T}$ and $\ddot{T}$ are readily available scalar quantities. The quantity $\nabla^T_y \dot{T}\,[X^p(x,y),y]$ is the true partial derivative unlike in (13b). This is easy to compute because y enters the $\dot{T}$ argument directly, at most, in two nonzero components. The expression for $\nabla_x X^p(x,y)$ has been derived already in (12a). The only remaining terms are $\nabla^2_{yx} X^p(x,y)$ which is built up inductively, and $\nabla^T_y X^p(x,y)$ which is discussed below.

Now

$$\nabla_y X^i(x,y) = D_y\ T[X^p(x,y),y]$$
$$= \nabla_y\ T[X^p(x,y),y] + \dot{T} \cdot \nabla_y X^p(x,y)\ . \tag{16}$$

Note $\nabla_y T(X^p(x,y),y)$ is the true partial derivative and will have at most two nonzero components in E^k which are easily computable. Thus (16) permits the computation of the $\nabla_y^T X^p(x,y)$ term of (14) by induction. Since the formulae of (12b), (14), (16) are initially applied to the original problem variables $X_i = x^i$, $i = 1,\ldots,n$, we have, for $p \leq n$:

$$\nabla_x X^i(x,y) = e_p \cdot \dot{T} \tag{17}$$

where e_j is a unit vector $\varepsilon\ E^n$ with unity in the jth position.

Also,

$$\nabla_{yx}^2 X^i(x,y) = \nabla_{yx}^2 X^p + e_p[\ddot{T}\ \nabla^T X^p + \nabla_y^T\ \dot{T}]$$
$$= e_p\ \nabla_y^T\ \dot{T}\ . \tag{18}$$

Finally,

$$\nabla_y X^i(x,y) = \nabla_y T + \nabla_y X^p\ \dot{T}$$
$$= \nabla_y T\ . \tag{19}$$

On page 6 , we spoke of dimensionally reducing the matrix of (1) in two separate ways. The case of premultiplying by $\alpha\ \varepsilon\ E^n$ is considered below.

Thus, from (17) and (18) letting α_p denote the pth component of α , we get

$$\alpha^T \nabla_x X^i(x,y) = \alpha_p \cdot \dot{T}\ . \tag{20}$$

$$\alpha^T \nabla_{yx}^2 X^i(x,y) = \alpha_p \nabla_y^T\ \dot{T}\ . \tag{21}$$

The summation of (21) will have, at most, two nonzero components due to the nature of the simple transformations used. Next, consider the case of $p > n$. Here,

$$\alpha^T \nabla_x X^i(x,y) = \alpha^T \nabla_x X^p(x,y) \cdot \dot{T} \cdot \tag{22}$$

Next,

$$\alpha^T \nabla^2_{yx} X^i(x,y) = \alpha^T \nabla^2_{yx} X^p(x,y) \cdot \dot{T}$$

$$+ \alpha^T \nabla_x X^p(x,y) \left\{ \ddot{T} \nabla_y^T X^p(x,y) + \nabla_y^T \dot{T} \right\} . \tag{23}$$

Equation (22) works purely inductively and does not require the deliberate introduction of α , as this has been done in (20). It is a scalar quantity for each i . The first term on the right-hand side of (23) is also an inductive term. As far as the second term is concerned, (the scalar multiplier $\alpha^T \nabla_x X^p(x,y)$, the only presence of α there,) it is built up inductively. The expression within the curly bracket is a k-vector, the first term being an inductive one as demonstrated in (16) and the second possessing, at most, two nonzero components.

In summary, we have a scalar quantity for the reduced version of the gradient vector of each CVF i , and (20) (or (17), if $p \leq n$) has to be repeated for every p corresponding to the separable terms of X^i , and added together. The reduced second cross partial of each CVF i (as shown in (21) or (18) depending on whether $p \gtrless n$) is a k-vector for each component p , which must be added together as before.

The quadratic components of (9) will be addressed next. Considering the creation of an X^i , $i = n+1,\ldots,N$, using a single product, we have

$$X^i(x,y) = V[X^p(x,y),y] * U[X^q(x,y),y] \ , \tag{24}$$

then,

$$\nabla_x X^i(x,y) = D_x V[X^p(x,y),y] * U[X^q(x,y),y]$$

$$+ V[X^p(x,y),y] * D_x U[X^q(x,y),y] \tag{25}$$

$$= \dot{V}U \cdot \nabla_x X^p(x,y) + V\dot{U} \cdot \nabla_x X^q(x,y) \ . \tag{26}$$

Now $\dot{V}$, $\dot{U}$ are defined analogously to $\dot{T}$ (see (11)). Since V , $\dot{V}$, U , $\dot{U}$ are scalar quantities, the premultiplication by $\alpha \in E^n$ could be done as in (20) and (22) depending on whether p and q are $\leq$ or $>$ n . The separate terms in (25) must be added together. The result is

$$\alpha^T \nabla_x X^i(x,y) = \dot{V}U\ \alpha^T \nabla_x X^p(x,y) + V\dot{U}\ \alpha^T \nabla_x X^q(x,y) \ . \tag{26}$$

When p (or q) is $> n$, $\alpha^T \nabla_x X^p$ (or $\alpha^T \nabla_x X^q$) enters purely inductively. Otherwise, the term simplifies to $\dot{V}U\ \alpha_p$ (or $V\dot{U}\ \alpha_q$). The second cross partial derivative is obtained by differentiating (26) with respect to y . The arguments of X^p and X^q are omitted for convenience of representation. Finally,

$$\begin{aligned} \nabla^2_{yx} X^i(x,y) = U &\left\{ \dot{V} \cdot \nabla^2_{yx} X^p + \nabla_x X^p [\ddot{V} \cdot \nabla^T_y X^p + \nabla^T_y \dot{V}] \right\} \\ &+ \dot{V} \cdot \nabla_x X^p \cdot D_y U \\ &+ V \left\{ \dot{U} \cdot \nabla^2_{yx} X^q + \nabla_x X^q [\ddot{U} \cdot \nabla^T_y X^q + \nabla^T_y \dot{U}] \right\} \\ &+ \dot{U} \cdot \nabla_x X^q \cdot D_y V \ . \end{aligned} \tag{27}$$

Premultiplying (27) by $\alpha \in E^n$, and rearranging,

$$\begin{aligned} \alpha^T \nabla^2_{yx} X^i(x,y) = U &\left\{ \alpha^T \nabla^2_{yx} X^p \cdot \dot{V} + \alpha^T \nabla_x X^p [\ddot{V} \cdot \nabla^T_y X^p + \nabla^T_y \dot{V}] \right\} \\ &+ V \left\{ \alpha^T \nabla^2_{yx} X^q \cdot \dot{U} + \alpha^T \nabla_x X^q [\ddot{U} \cdot \nabla^T_y \dot{U}] \right\} \\ &+ \alpha^T \nabla_x X^p \cdot D_y U \cdot \dot{V} + \alpha^T \nabla_x X^q \cdot \nabla_y D \cdot \dot{U} \ . \end{aligned} \tag{28}$$

The evaluation of the portions within the curly brackets in (28) is identical to the evaluation of the cross second partials in the separable analysis, i.e., the right-hand side of (23). The last two terms of (26) contain two quantities $\alpha^T \nabla_x X^p$ and $\alpha^T \nabla_x X^q$ which are developed by induction as demonstrated by (12) and (20). The quantities $D_y U$ and $D_y V$

are analogous to $D_y T$ whose computation is shown in (16). The only remaining expression to be developed is that for $\nabla_y X^i$,

$$\begin{aligned} \nabla_y X^i(x,y) &= D_y\{V[X^p(x,y),y]\ U[X^q(x,y),y]\} \\ &= U\{\dot{V}\cdot\nabla_y X^p+\nabla_y V\} + V\{\dot{U}\ \nabla_y X^q+\nabla_y U\} \ . \end{aligned} \tag{29}$$

Examination of (23) and (28) shows that it is not necessary to premultiply the expressions of (16) and (29) by α^T , since $\nabla_y X$ never gets directly premultiplied by it.

The above Equations (20) through (29) have developed the necessary components to perform the computations necessary to produce a row k-vector $\alpha^T[dx(y)/dy]$. The next section describes how these are implemented via a penalty function method.

4. Computation of Sensitivity Results

The SUMT computer code [4] is used in conjunction with the factorable symbolic sensitivity analysis code to yield $[\alpha^T dx(y)/dy]$. The code uses an interior-exterior penalty function with a single parameter r , and generates a sequence of subproblems which will be denoted as $M(y,r)$, which tend to $M(y)$ as $r \to 0$. In straight nonlinear optimization SUMT uses an extrapolation based on r to estimate the relevant values at $r = 0$. A similar extrapolation technique is used here to provide first order estimates of $[\alpha^T dx(y)/dy]$.

A logarithmic barrier quadratic loss penalty function is considered below. For $M(y)$, this penalty function is

$$P[x,y,r] = F(x,y) - r \sum_{i=1}^{m} \ln G_i(x,y) + \sum_{j=1}^{p} \frac{H_j^2(x,y)}{r} \ . \tag{30}$$

The problem solved at the ℓth stage of the sequential procedure, for a value $r = r^\ell$ is

$$\min_{x\in R^0} P[x,y,r^\ell] \qquad \text{Problem } M(r,y)$$

where

$$R^0 \equiv \{x \mid G_i(x,y) > 0\ ,\ i=1,\dots,m\} \ .$$

The optimality condition is that the gradient with respect to x vanishes, i.e.,

$$\nabla_x P[x,y,r^\ell] = 0 . \tag{31}$$

At local optima, for a general value of r we can write

$$\nabla_x P[x(y,r),y,r] \equiv 0 . \tag{32}$$

Applying this to (30),

$$\nabla_x F[x(y,r),y] - \sum_{i=1}^{m} \frac{r}{G_i[x(y,r),y]} \cdot \nabla_x G_i[x(y,r),y] + \sum_{i=1}^{p} \frac{2\,H_j[x(y,r),y]}{r} \nabla_x H_j[x(y,r),y] \equiv 0 . \tag{33}$$

Differentiating (33) partially with respect to y , we get

$$\begin{aligned} \nabla^2_{yx} P[x(y,r),y,r] &= \nabla^2_{yx} F[x(y,r),y] - \sum_{i=1}^{m} \nabla^2_{yx} G_i[x(y,r),y] \cdot \frac{r}{G_i[x(y,r),y]} \\ &+ \sum_{i=1}^{m} \frac{r}{G_i[x(y,r),y]^2} \cdot \nabla_x G_i[x(y,r),y] \cdot \nabla^T_y G_i[x(y,r),y] \\ &+ \sum_{j=1}^{p} \nabla^2_{yx} H_j[x(y,r),y] \cdot \frac{2\,H_j[x(y,r),y]}{r} \\ &+ \sum_{j=1}^{p} \left(\frac{2}{r}\right) \nabla_x H_j[x(y,r),y] \cdot \nabla^T_y H_j[x(y,r),y] . \end{aligned} \tag{34}$$

Differentiating (33) totally with respect to y ,

$$\nabla^2_{xx} P[x(y,r),y,r] \frac{dx(y,r)}{dy} + \nabla^2_{yx} P[x(y,r),y,r] \equiv 0 . \tag{35}$$

The argument variables within the square brackets will be omitted when referring to the terms in the above.

$\nabla^2_{xx}P$ has been developed elsewhere by McCormick [5]. Those for $\nabla^2_{xy}P$ are developed using the Equation (34). Thus,

$$\frac{d(x(y,r))}{dy} = - \nabla^2_{xx}P[x(y,r),y,r]^{-1} \cdot \nabla^2_{yx}P[x(y,r),y,r] \ . \tag{36}$$

The similarity between (7) and (36) is to be noted. Suppose (36) is premultiplied by some $\sigma \varepsilon E^n$. Then, we get

$$\sigma^T \frac{dx(y,r)}{dy} = - \alpha^T \cdot \nabla^2_{yx}P[x(y,r),y,r] \ , \tag{37}$$

where

$$\alpha^T = \sigma^T \cdot \nabla^2_{xx}P[x(y,r),y,r]^{-1} \ . \tag{38}$$

If σ is a known vector, α is determined because the Hessian of the penalty function with respect to x is known by [5]. Then (37) is determined by premultiplying (34) by this α , and using (20), (21), (22), (23), (27) and (29) to determine the individual components of the resulting equation. This can be done because every problem function F , G_i ; $i = 1,\ldots,m$, H_j ; $j = 1,\ldots p$, can be identified with a CVF X^i , $i = 1,\ldots,n$, $n+1,\ldots,N$.

The final aspect remaining to be discussed in the context of the sensitivity methodology is the improvement of the estimate of $\sigma^T \frac{dx(y,r)}{dy}$ by extrapolating on r , at the ℓth sequential unconstrained problem, to $r = 0$. A Taylor series approximation applied at the ℓth and $(\ell-1)$th subproblems yields

$$\frac{dx(y,r^{\ell})}{dy} \doteq \frac{dx(y,0)}{dy} + r^{\ell}\,\frac{d}{dr}\left[\frac{dx(y,0)}{dy}\right]. \tag{39}$$

$$\frac{dx(y,r^{\ell-1})}{dy} \doteq \frac{dx(y,0)}{dy} + r^{\ell-1}\,\frac{d}{dr}\left[\frac{dx(y,0)}{dy}\right]. \tag{40}$$

Eliminating the last term in (39) and (40)

$$\frac{d(x(y,0))}{dy}\left(r^{\ell-1} - r^{\ell}\right) \doteq r^{\ell-1}\cdot\frac{dx(y,r^{\ell})}{dy} - r^{\ell}\cdot\frac{dx(y,r^{\ell-1})}{dy} \tag{41}$$

from which, (letting $r^{\ell-1} = c \cdot r^{\ell}$) we get

$$\frac{dx(y,0)}{dy} \doteq \frac{1}{c-1}\cdot\left[c\,\frac{dx(y_1 r^{\ell})}{dy} - \frac{dx(y_1 r^{\ell-1})}{dy}\right]. \tag{42}$$

Hence by performing the computation for successive subproblems when r becomes sufficiently small, (42) can be used to estimate the value for the original problem, i.e., $r = 0$.

REFERENCES

[1] ARMACOST, R. L. and W. C. MYLANDER (1973). A guide to a SUMT-Version 4 computer subroutine for implementing sensitivity analysis in nonlinear programming. Technical Paper Serial T-237. The Institute for Management Science and Engineering, The George Washington University.

[2] FIACCO, A. V. (1976). Sensitivity analysis for nonlinear programming using penalty methods. Mathematical Programming 10 (3) 287-311.

[3] FIACCO, A. V. and G. P. MCCORMICK (1968). Nonlinear Programming: Sequential Unconstrained Minimization Techniques. Wiley, New York.

[4] HOLMES, R. L., W. C. MYLANDER, and G. P. MCCORMICK (1971). A guide to SUMT-Version 4: The computer program implementing the sequential unconstrained minimization technique for nonlinear programming. RAC-P-63, Research Analysis Corporation, McLean, Virginia.

[5] MCCORMICK, G. P. (1974). A mini-manual for use of the SUMT computer program and the factorable programming language. Technical Report SOL 74-15. Systems Optimization Laboratory, Stanford University.

[6] ORTEGA, J. M. and W. C. RHEINBOLDT (1970). Iterative Solutions of Nonlinear Equations in Several Variables. Academic Press.

THE GEORGE WASHINGTON UNIVERSITY
Program in Logistics
Distribution List for Technical Papers

The George Washington University
  Office of Sponsored Research
  Library
  Vice President H. F. Bright
  Dean Harold Liebowitz
  Dean Henry Solomon

ONR
  Chief of Naval Research
    (Codes 200, 434)
  Resident Representative

OPNAV
  OP-40
  DCNO, Logistics
  Navy Dept Library
  NAVDATA Automation Cmd
  OP-964

Naval Aviation Integrated Log Support

NARDAC Tech Library

Naval Electronics Lab Library

Naval Facilities Eng Cmd Tech Library

Naval Ordnance Station
  Louisville, Ky.
  Indian Head, Md.

Naval Ordnance Sys Cmd Library

Naval Research Branch Office
  Boston
  Chicago
  New York
  Pasadena
  San Francisco

Naval Ship Eng Center
  Philadelphia, Pa.
  Washington, DC

Naval Ship Res & Dev Center

Naval Sea Systems Command
  PMS 30611
  Tech Library
  Code 073

Naval Supply Systems Command
  Library
  Operations and Inventory Analysis

Naval War College Library
  Newport

BUPERS Tech Library

FMSO

Integrated Sea Lift Study

USN Ammo Depot Earle

USN Postgrad School Monterey
  Library
  Dr Jack R. Borsting
  Prof C. R. Jones

US Marine Corps
  Commandant
  Deputy Chief of Staff, R&D

Marine Corps School Quantico
  Landing Force Dev Ctr
  Logistics Officer
  Commanding Officer
    USS Francis Marion (LPA-249)

Armed Forces Industrial College

Armed Forces Staff College

Army War College Library
  Carlisle Barracks

Army Cmd & Gen Staff College

Army Logistics Mgt Center
  Fort Lee

Commanding Officer, USALDSRA
  New Cumberland Army Depot

Army Inventory Res Ofc
  Philadelphia

Air Force Headquarters
  AFADS-3
  LEXY
  SAF/ALG

Griffiss Air Force Base
  Reliability Analysis Center

Gunter Air Force Base
  AFLMC/XR

Maxwell Air Force Base Library

Wright-Patterson Air Force Base
  Log Command
  Research Sch Log
  AFALD/XR

Defense Documentation Center

National Academy of Sciences
  Maritime Transportation Res Board Library

National Bureau of Standards
  Dr B. H. Colvin
  Dr Joan Rosenblatt

National Science Foundation

National Security Agency

Weapon Systems Evaluation Group

British Navy Staff

National Defense Hdqtrs, Ottawa
  Logistics, OR Analysis Establishment

American Power Jet Co
  George Chernowitz

General Dynamics, Pomona

General Research Corp
  Dr Hugh Cole
  Library

Logistics Management Institute
  Dr Murray A. Geisler

MATHTEC
  Dr Eliot Feldman

Rand Corporation
  Library

Carnegie-Mellon University
  Dean H. A. Simon
  Prof G. Thompson

Case Western Reserve University
  Prof B. V. Dean
  Prof M. Mesarovic
  Prof S. Zacks

Cornell University
  Prof R. E. Bechhofer
  Prof R. W. Conway
  Prof Andrew Schultz, Jr.

Cowles Foundation for Research in Economics
  Prof Herbert Scarf
  Prof Martin Shubik

Florida State University
  Prof R. A. Bradley

Harvard University
  Prof K. J. Arrow
  Prof W. G. Cochran
  Prof Arthur Schleifer, Jr.

Princeton University
  Prof A. W. Tucker
  Prof J. W. Tukey
  Prof Geoffrey S. Watson

Purdue University
Prof S. S. Gupta
Prof H. Rubin
Prof Andrew Whinston

Stanford University
Prof T. W. Anderson
Prof G. B. Dantzig
Prof F. S. Hillier
Prof D. L. Iglehart
Prof Samuel Karlin
Prof G. J. Lieberman
Prof Herbert Solomon
Prof A. F. Veinott, Jr.

University of California, Berkeley
Prof R. E. Barlow
Prof D. Gale
Prof Jack Kiefer
Prof Rosedith Sitgreaves

University of California, Los Angeles
Prof J. R. Jackson
Prof R. R. O'Neill

University of North Carolina
Prof W. L. Smith
Prof M. R. Leadbetter

University of Pennsylvania
Prof Russell Ackoff
Prof Thomas L. Saaty

University of Texas
Prof A. Charnes

Yale University
Prof F. J. Anscombe
Prof I. R. Savage

Prof Z. W. Birnbaum
University of Washington

Prof B. H. Bissinger
The Pennsylvania State University

Prof Seth Bonder
University of Michigan

Prof G. E. P. Box
University of Wisconsin

Dr Jerome Bracken
Institute for Defense Analyses

Prof H. Chernoff
Mass. Institute of Technology

Prof Arthur Cohen
Rutgers - The State University

Mr Wallace M. Cohen
US General Accounting Office

Prof C. Derman
Columbia University

Prof Masao Fukushima
Kyoto University

Prof Saul I. Gass
University of Maryland

Dr Donald P. Gaver
Carmel, California

Prof Amrit L. Goel
Syracuse University

Prof J. F. Hannan
Michigan State University

Prof H. O. Hartley
Texas A & M Foundation

Mr Gerald F. Hein
NASA, Lewis Research Center

Prof W. M. Hirsch
Courant Institute

Dr Alan '. Hoffman
IBM, Yorktown Heights

Prof John R. Isbell
State University of New York, Amherst

Dr J. L. Jain
University of Delhi

Prof J. H. K. Kao
Polytech Institute of New York

Prof W. Kruskal
University of Chicago

Mr S. Kumar
University of Madras

Prof C. E. Lemke
Rensselaer Polytech Institute

Prof Loynes
University of Sheffield, England

Prof Steven Nahmias
University of Pittsburgh

Prof D. B. Owen
Southern Methodist University

Prof E. Parzen
Texas A & M University

Prof H. O. Posten
University of Connecticut

Prof R. Remage, Jr.
University of Delaware

Prof Hans Riedwyl
University of Bern

Dr Fred Rigby
Texas Tech College

Mr David Rosenblatt
Washington, D. C.

Prof M. Rosenblatt
University of California, San Diego

Prof Alan J. Rowe
University of Southern California

Prof A. H. Rubenstein
Northwestern University

Dr M. E. Salveson
West Los Angeles

Prof Edward A. Silver
University of Waterloo, Canada

Prof M. J. Sobel
Georgia Inst of Technology

Prof R. M. Thrall
Rice University

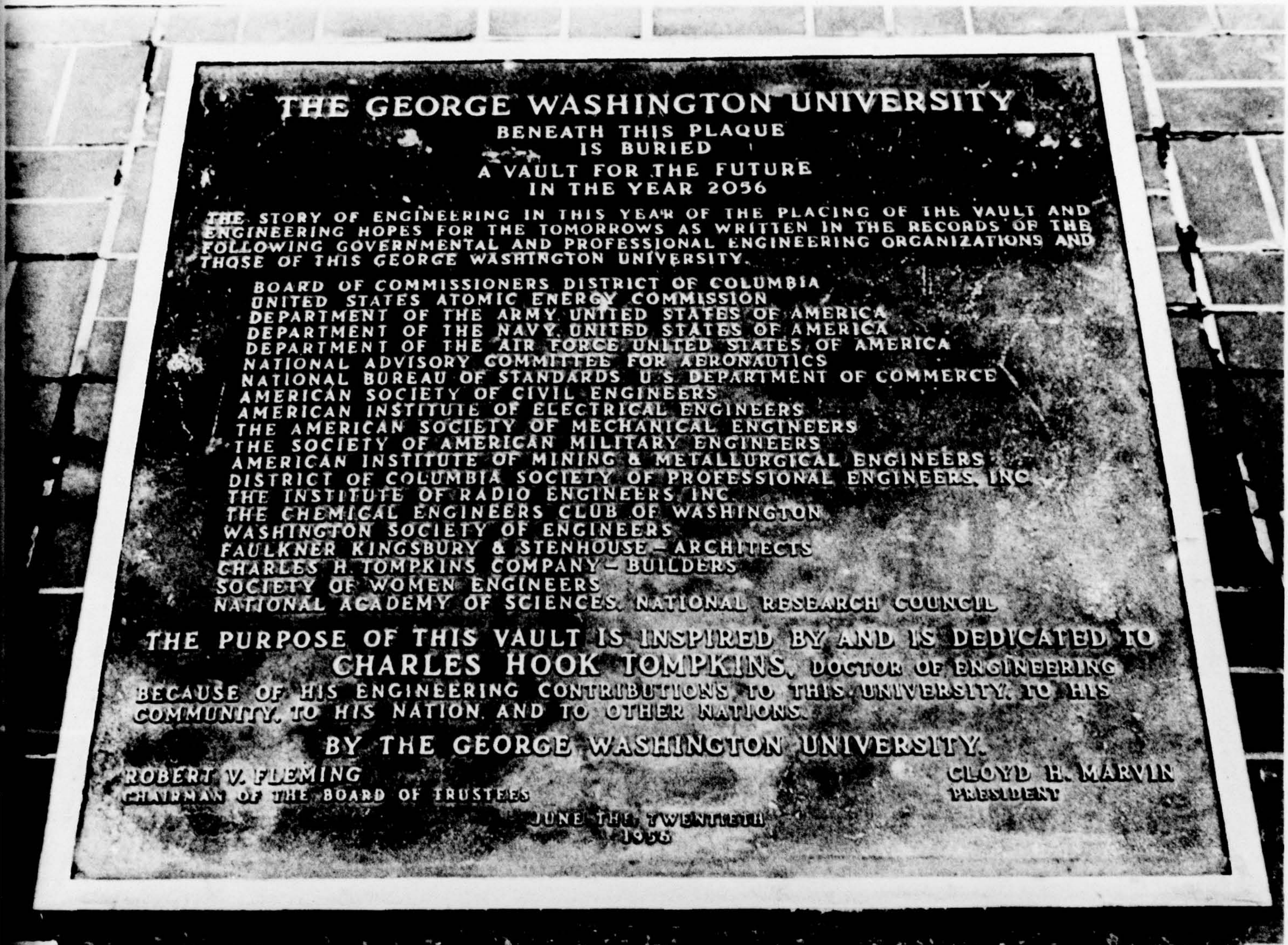
Dr S. Vajda
University of Sussex, England

Prof T. M. Whitin
Wesleyan University

Prof Jacob Wolfowitz
University of South Florida

Prof Max A. Woodbury
Duke University

December 1978



To cope with the expanding technology, our society must be assured of a continuing supply of rigorously trained and educated engineers. The School of Engineering and Applied Science is completely committed to this objective.